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# An Approximate Analytical Solution for Default-Free Zero-Coupon Bonds in a Two-Factor Vasicek Interest Rate Model with Stochastic Volatility

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**Abstract:** ***To address the limitation of the traditional Vasicek interest rate model, in which volatility is assumed to be constant and therefore cannot adequately describe the stochastic evolution of interest rate volatility, this paper introduces a stochastic volatility factor into a mean-reverting short-rate framework. A two-factor Vasicek-type interest rate model is developed for the approximate analytical pricing of default-free zero-coupon bonds. By applying the integrating factor method, the short-rate process is represented in integral form, and the bond price is transformed into the conditional expectation of a discount factor generated by the future short-rate integral. Conditional on the volatility path, the conditional Gaussian property of the stochastic integral is used to derive a moment-generating-function representation. Following the stochastic-volatility approximation idea of Hull and White, the pricing expression is further approximated through the time average of the squared volatility process and a third-order Taylor expansion. The resulting formula contains correction terms determined by the variance and the third central moment of the average variance. The proposed method preserves the main effects of stochastic volatility while maintaining analytical tractability and computational efficiency, thereby providing a useful approximation framework for pricing zero-coupon bonds and related interest rate derivatives.***

**Keywords:** Vasicek model, Stochastic volatility, Zero-coupon bond, Approximate analytical solution, Taylor expansion, Term structure of interest rates.

## 1. Introduction

Term structure models of interest rates are fundamental to fixed-income pricing, interest rate risk management, and financial engineering. As a basic fixed-income instrument, the default-free zero-coupon bond provides the foundation for pricing more complex interest rate derivatives, including bond options, swaps, caps, and floors. The classical Vasicek model describes the short rate through a mean-reverting process and yields a tractable analytical formula for zero-coupon bond pricing [1].

However, the standard Vasicek model assumes constant short-rate volatility, which limits its ability to capture the time-varying, stochastic, and clustering features of interest rate volatility observed in financial markets. During changes in macroeconomic conditions, monetary policy adjustments, liquidity shocks, or extreme events, interest rate volatility may fluctuate significantly. Using a constant-volatility model in such settings may lead to valuation bias and may affect hedging and risk-measurement accuracy [4][5].

To address this limitation, stochastic volatility models have been widely used in asset and derivative pricing. Hull and White proposed an approximate analytical approach based on a Taylor expansion with respect to average variance[3].

Approximate analytical methods are particularly useful when closed-form solutions are unavailable. Techniques such as Taylor expansion, moment matching, characteristic functions, and Fourier transforms can reduce computational costs while preserving key model features. Lin and Chang applied this idea to Asian option pricing under stochastic volatility and derived an efficient approximate analytical formula, showing that low-order moment approximations can effectively capture the main effects of stochastic volatility[6].

Motivated by these studies, this paper introduces a stochastic volatility factor into the classical Vasicek short-rate model and constructs a two-factor interest rate model for pricing default-free zero-coupon bonds. The proposed model preserves the mean-reverting structure of the short rate while allowing its diffusion term to vary stochastically. Since the resulting short-rate integral no longer has a simple normal distribution, this paper applies the integrating factor method, conditional Gaussian arguments, and the moment-generating-function approach to express the bond price in terms of an average variance. A third-order Taylor expansion is then used to obtain an approximate analytical pricing formula.

The contributions of this paper are threefold. First, it extends the Vasicek framework by incorporating stochastic volatility. Second, it derives a tractable approximate analytical formula for default-free zero-coupon bond pricing. Third, it applies the Taylor expansion technique from stochastic-volatility option pricing to fixed-income valuation, thereby providing a basis for further research on bond options and other interest rate derivatives.

## 2. 2. Model Setup and Conditional Gaussian Representation

Assume that the financial market is frictionless and arbitrage-free, and that pricing is conducted under the risk-neutral measure $Q$. Let $r(t)$ denote the short rate at time $t$, and let $\sigma(t)$ denote the stochastic volatility factor of the short rate. This paper considers the following two-factor Vasicek-type interest rate model with stochastic volatility:

$$\begin{cases} dr(t) = [\beta - \alpha r(t)]dt + \sigma(t)dW_t \\ d\sigma(t) = k\sigma(t)dt + \gamma\sigma(t)dZ_t \end{cases}.$$

$$d < W, Z >_t = \rho dt.$$

Here, $\alpha > 0$ denotes the mean-reversion speed, and $\beta > 0$determines the long-run mean level $\frac{\beta}{\alpha}$ of the short rate. The parameters $\kappa$ and $\gamma$ denote the drift and diffusion parameters of the volatility factor, respectively. $W_t$ and $Z_t$ are Brownian motions, and $\rho$ is their correlation coefficient.

For analytical tractability, the subsequent derivation focuses on the uncorrelated benchmark case, namely $\rho = 0$. The correlated case is left for future research. When $\sigma(t)$ degenerates into a constant, the model reduces to the classical Vasicek model.

The price at time $t$ of a default-free zero-coupon bond with maturity $T$ is defined as the risk-neutral conditional expectation of the discount factor associated with the future integral of the short rate:

$$B(t,T) = E_t{}^Q\left[e^{-\int_t^T r(u)du}\right].$$

Let $\tau = T - t$. By applying the integrating factor method to the linear short-rate equation, the time integral of the short rate over $[t, T]$ can be written as:

$$\int_t^T r(u)du = A(\tau) + B(\tau)r(t) + \int_t^T G(s,T)\,\sigma(s)dW_s.$$

Where $B(\tau) = \frac{1-e^{-\alpha\tau}}{\alpha}$, $A(\tau) = \frac{\beta}{\alpha}(\tau - B(\tau))$,

$$G(s,T) = \frac{1-e^{-\alpha(T-s)}}{\alpha}\ , s \in [t,T].$$

Therefore, the zero-coupon bond price can be written as

$$B(t,T) = \exp\left(-A(\tau) - B(\tau)r(t)\right)E_t{}^Q\left[\exp\left(-\int_t^T G(s,T)\,\sigma(s)dW_s\right)\right].$$

Define $X = -\int_t^T G(s,T)\,\sigma(s)dW_s$.

Conditional on the volatility path $\sigma(u), u \in [t,T]$, $G(s,T)\sigma(s)$can be regarded as a deterministic function of the time variable $s$. Hence, the stochastic integral $X$ has a conditional normal structure:

$$X|\sigma \sim N\left(0, \int_t^T [G(s,T)\sigma(s)]^2\,ds.\right.$$

$$E_t{}^Q[X|\sigma] = 0,$$

$$var[X|\sigma] = \int_t^T [G(s,T)\sigma(s)]^2\,ds.$$

By the moment-generating function of a normally distributed random variable,

$$E_t{}^Q[e^X|\sigma] = \exp\left(\frac{1}{2}\int_t^T [G(s,T)\sigma(s)]^2\,ds\right).$$

Let $V(s) = \sigma^2(s)$. Thus,

$$B(t,T) = \exp\left(-A(\tau) - B(\tau)r(t)\right)E_t{}^Q\left[\exp\left(\frac{1}{2}\int_t^T [G(s,T)]^2\,V(s)ds\right)\right].$$

For the convenience of subsequent approximation analysis, define the deterministic weight integral and the weighted average variance as

$$H(t,T) = \int_t^T (G(s,T))^2\,ds,$$

$$\overline{V_w}(t,T) = \frac{\int_t^T [G(s,T)]^2 V(s)ds}{H(t,T)}.$$

It follows that

$$\int_t^T G(s,T)^2\,V(s)ds = H(t,T)\overline{V_w}(t,T).$$

Consequently, the bond price can be rewritten as

$$B(t,T) = \exp\left(-A(\tau) - B(\tau)r(t)\right)E_t{}^Q\left[\exp\left(\frac{1}{2}H(t,T)\overline{V_w}(t,T)\right)\right].$$

The expression above shows that stochastic volatility enters the pricing formula through the weighted average variance. Since the higher-order moments of $\overline{V_w}(t,T)$, which is weighted by $G(s,t)^2$, are generally difficult to obtain in closed form, we adopt a Hull-White-type approximation and replace the weighted average variance by the ordinary time-average variance:

$$\overline{V_w}(t,T) \approx \hat{v}(t),\ \hat{v} = \frac{1}{\tau}\int_t^T V(u)du.$$

Accordingly,

$$B(t,T) \approx \exp\left(-A(\tau) - B(\tau)r(t)\right)E_t{}^Q\left[\exp\left(\frac{1}{2}H(t,T)\hat{v}(t)\right)\right].$$

## 3. Approximate Analytical Result for Default-Free Zero-Coupon Bonds

This section derives a tractable approximation by applying a Taylor expansion to the bond price function driven by the ordinary time-average variance. Let

$$C(\bar{v}) = \exp\left[-A(\tau) - B(\tau)r(t) + \frac{1}{2}H(t,T)\bar{v}\right].$$

Then the approximate bond price can be written as

$$B(t,T) \approx E_t{}^Q[C(\bar{v})].$$

To obtain analytical expressions for the low-order moments of $\bar{v}$, consider the squared volatility process $V(t) = \sigma^2(t)$. By Ito's lemma,

$$dV(t) = (2k + \gamma^2)V(t)dt + 2\gamma V(t)dZ_t.$$

For analytical tractability, this paper considers the martingale case $2k + \gamma^2 = 0$.

In this case,

$$dV(t) = \xi_V V(t)dZ_t(u), \xi_V = 2\gamma.$$

and $V(t)$ is a geometric Brownian martingale. Define the ordinary time-average variance over $[t, T]$ by

$$\hat{v} = \frac{1}{\tau}\int_t^T V(s)ds\,, \tau = T - t.$$

Following the moment calculation approach of Hull and White, the first three raw moments of $\hat{v}(t)$ are

$$E_t{}^Q[\hat{v}] = V(t),$$

$$E_t{}^Q[\hat{v}^2] = \frac{2V(t)^2(e^{K_V} - K_V - 1)}{K_V{}^2}\ , K_V = \xi_V{}^2\tau,$$

$$E_t{}^Q[\hat{v}^3] = \frac{V(t)^3(e^{3K_V} - 9e^{K_V} + 6K_V + 8)}{3K_V{}^3}, K_V = \xi_V{}^2\tau.$$

Hence, the variance and the third central moment of the average variance are

$$Var_t[\hat{v}] = E_t{}^Q[\hat{v}^2] - \left(E_t{}^Q[\hat{v}]\right)^2,$$

$$Skewt = E_t{}^Q[(\hat{v} - E_t[\hat{v}])^3]$$
$$= E_t{}^Q[\hat{v}^3] - 3E_t{}^Q[\hat{v}^2]E_t[\hat{v}] + 2\left(E_t{}^Q[\hat{v}]\right)^3$$

Expanding $C(\bar{v})$ around $\hat{v} = \mathbb{E}_t[\bar{V}] = V(t)$ up to the third order and taking conditional expectations gives

$$E_t{}^Q[C(V)] \approx C\big(V(t)\big) + \tfrac{1}{2}C''\big(V(t)\big)Var_t[\hat{v}] + \tfrac{1}{6}C'''(V(t))Skewt.$$

Since

$$C''\big(V(t)\big) = \left[\tfrac{H(t,T)^2}{4}\right]C\big(V(t)\big),$$
$$C'''\big(V(t)\big) = \left[\tfrac{H(t,T)^3}{8}\right]C\big(V(t)\big).$$

the approximate analytical pricing formula is

$$B(t,T) \approx \exp\left(-\mathrm{A}(\tau) - \mathrm{B}(\tau)\mathrm{r}(\mathrm{t}) + \frac{1}{2}H(t,T)V(t)\right) \times [1 + \tfrac{H(t,T)^2}{8}Var_t[\hat{v}] + \tfrac{H(t,T)^3}{48}Skewt].$$

Equivalently, $H(t,T)$ can be evaluated explicitly as

$$H(t,T) = \tfrac{1}{\alpha^2}[\tau - 2\left(\tfrac{1-e^{-\alpha\tau}}{\alpha}\right) + \tfrac{1-e^{-2\alpha\tau}}{2\alpha}].$$

The first term in the pricing formula corresponds to the benchmark price obtained by replacing the random average variance with its conditional mean. The second and third terms represent the correction effects induced by the variance and skewness of the average variance, respectively.

## 4. Conclusion

This paper investigates the approximate analytical pricing of default-free zero-coupon bonds under a two-factor Vasicek interest rate model with a stochastic volatility factor. Although the traditional Vasicek model has a mean-reverting structure and strong analytical tractability, its constant-volatility assumption cannot adequately capture the stochastic variation of volatility observed in real interest rate markets. To address this limitation, this paper introduces a stochastic volatility factor into the mean-reverting short-rate equation and constructs a two-factor model capable of jointly describing the dynamics of both the interest rate level and interest rate volatility.

From the perspective of theoretical derivation, this paper first applies the integrating factor method to obtain the short-rate process and its time integral, and then represents the zero-coupon bond price as the conditional expectation of the discount factor generated by the future short-rate integral. Conditional on the volatility path, the conditional Gaussian structure of the stochastic integral is used to transform the bond price into a function of the weighted average variance. When the squared volatility process satisfies the martingale condition, the first three moments of the average variance can be obtained by following the moment calculation method of Hull and White, and an approximate analytical pricing formula can be derived through a third-order Taylor expansion.

The resulting formula shows that the effect of stochastic volatility on zero-coupon bond prices is reflected not only in the average volatility level, but also in the correction terms induced by the variance and the third central moment of the average variance. Compared with the traditional constant-volatility Vasicek model, the proposed model can therefore more fully capture the influence of volatility uncertainty on the value of fixed-income products. The method has a clear analytical structure, relatively high computational efficiency, and convenient applicability to parameter sensitivity analysis. It may thus provide a useful reference for bond pricing and risk measurement under stochastic-volatility interest rate models.

Future research may be extended in three directions. First, based on the approximate analytical solution for zero-coupon bonds developed in this paper, one may further study the pricing of European bond options written on zero-coupon bonds and employ Fourier transform methods to handle nonlinear payoff functions. Second, the correlation structure between the Brownian motion driving the short rate and that driving the volatility process can be characterized more explicitly, so as to derive more complete correlation correction terms. Third, the model may be extended to jump-diffusion settings, stochastic mean-reversion levels, or multifactor term structure models, and its pricing accuracy and practical applicability may be examined through calibration to market data.

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## Appendix A. Main Proof Procedures

### A.1 Integral Representation of the Short Rate

Applying the integrating factor $e^{\alpha t}$ to the short-rate equation yields

$$de^{\alpha t}r(t) = \beta e^{\alpha t}dt + e^{\alpha t}\sigma(t)dW_t.$$

Integrating over the interval $[t,u]$and rearranging terms, we obtain

$$r(u) = e^{-\alpha(u-t)}r(t) + \frac{\beta}{\alpha}\left(1 - e^{-\alpha(u-t)}\right) + \int_t^u e^{-\alpha(u-s)}\sigma(s)dW_s$$

Integrating with respect to $u$ from t to $T$ and applying Fubini's theorem to interchange the order of integration gives

$$\int_t^T \int_t^u e^{-\alpha(u-s)}\sigma(s)\, dW_s du = \int_t^T \left[\left[\int_s^T e^{-\alpha(u-s)} du\right]\sigma(s) dW_s\right.$$

it follows that

$$\int_t^T r(u)du = A(\tau) + B(\tau)r(t) + \int_t^T G(s,T)\sigma(s)dW_s.$$

where $B(\tau)$, $A(\tau)$, and $G(s,T)$ are defined in Section 2.

**A.2 Conditional Expectation and Moment-Generating Function**

Conditional on the volatility path, $G(s,T)\sigma(s)$ can be regarded as a deterministic integrand with respect to time. By the Gaussian property of the Ito integral,

$$X = -\int_t^T G(s,T)\, \sigma(s)dW_s.$$

is conditionally normally distributed with mean zero and variance $\int_t^T [G(s,T)\sigma(s)]^2\, ds$.
Therefore,

$$E_t[e^X|\sigma] = \exp\left(\frac{1}{2}\int_t^T [G(s,T)\sigma(s)]^2\, ds\right).$$

The zero-coupon bond price can therefore be written as

$$B(t,T) = \exp\left(-\mathrm{A}(\tau) - \mathrm{B}(\tau)\mathrm{r}(\mathrm{t})\right) \cdot E_t\left[\exp\left(\frac{1}{2}\int_t^T [G(s,T)\sigma(s)]^2\, ds\right)\right].$$

Introducing $H(t,T)$ and the weighted average variance $\overline{V_w}(t,T)$, we obtain

$$\int_t^T [G(s,T)\sigma(s)]^2\, ds = H(t,T) \cdot \overline{V_w}(t,T).$$

**A.3 Moment Calculation of the Average Variance**

Let $V(t) = \sigma^2(t)$. By Ito's lemma,

$$dV(t) = (2k + \gamma^2)V(t)dt + 2\gamma V(t)dZ_t.$$

When $2k + \gamma^2 = 0$ , $V(t)$ is a geometric Brownian martingale and

$$dV(t) = \xi_V V(t)dZ_t, \xi_V = 2\gamma.$$

Thus, for $s$ in $[t,T]$,

$$V(s) = V(t)exp(-\frac{1}{2}{\xi_V}^2(s-t) + \xi_V(Z_s - Z_t)).$$

By the moment formula for lognormal processes

$$E_t[V(s)] = V(t),$$
$$E_t[V(s_1)V(s_2)] = V(t)^2 \exp\left[{\xi_V}^2 \min(s_1 - t, s_2 - t)\right],$$
$$E_t[V(s_1)V(s_2)V(s_3)] = V(t)^3 \exp\left[{\xi_V}^2(min_{12} + min_{13} + min_{23})\right].$$

Where $min_{ij} = \min\,(s_i - t, s_j - t)$. Integrating the above expressions over $[t,T]$ gives the first three raw moments of the ordinary time-average variance $\bar{v}$ :

$$E_t[\hat{v}] = V(t),$$

$$E_t[\hat{v}^2] = \frac{2V(t)^2(e^{K_V} - K_V - 1)}{{K_V}^2},$$

$$E_t[\hat{v}^3] = \frac{V(t)^3(e^{3K_V} - 9e^{K_V} + 6K_V + 8)}{3{K_V}^3},$$

$$K_V = {\xi_V}^2\tau.$$

The variance and third central moment are then obtained from

$$Var_t[\hat{v}] = E_t[\hat{v}^2] - (E_t[\hat{v}])^2,$$
$$Skewt = E_t[\hat{v}^3] - 3E_t[\hat{v}^2]E_t[\hat{v}] + 2(E_t[\hat{v}])^3.$$