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A Note on Directed Divergence Measures

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Abstract: ***The measures of directed divergence of parametric entropy have been obtained which are generalizations of Shannon's Kapur's, Bose Einstein, Fermi-Dirac, and Havrda-Charvat's measures of Entropy. We have also examined its concavity property and some special cases.***

Keywords: Measure of Entropy

1. Introduction

(I) Measures of Directed divergence corresponding to the measure of entropy

$$H_{\frac{b}{a}}(P) = -\sum_{i=1}^{n} p_i ln p_i + \frac{a}{b}\sum_{i=1}^{n}\left(1+\frac{b}{a}p_i\right) ln\left(1+\frac{b}{a}p_i\right) - \frac{a}{b}\left(1+\frac{b}{a}\right) ln\left(1+\frac{b}{a}\right) \text{ , b> -1 a>0} \quad (1)$$

The proposed measure of directed divergence

$$D\,(P{:}\,Q) = \sum_{i=1}^{n} p_i ln\frac{p_i}{q_i} - \frac{a}{b}\sum_{i=1}^{n}(1+\frac{b}{a}p_i) ln\frac{(1+\frac{b}{a}p_i)}{(1+\frac{b}{a}p_i)} \quad \text{b> -1, a>0} \quad (2)$$

This measure holds all the properties

This is permutationally symmetric, Continuous convex function of $p_1, p_2, p_3, \ldots, p_n$ and vanishes iff $p_i = q_i$ $\forall\, i$

However, it is not in general a convex function of $q_1, q_2 \ldots q_n$

Now to generalized equation (2)

To consider the measure

$$D\,(P{:}\,Q) = \sum_{i=1}^{n} p_i ln\frac{p_i}{q_i} + A\sum_{i=1}^{n}(i+\frac{b}{a}p_i) ln\frac{(1+\frac{b}{a}p_i)}{1+\frac{b}{a}q_i} \quad (3)$$

This is convex function of $p_1, p_2, \ldots p_n$

If, $D'(P{:}\,Q) = ln\frac{P_i}{q_i} + 1 + A.\frac{b}{a} ln\frac{1+\frac{b}{a}P_i}{1+\frac{b}{a}q_i} + A.\frac{b}{a}$

$D''(P{:}\,Q) = \frac{1}{P_i} + A\frac{(\frac{b}{a})^2}{1+\frac{b}{a}P_i}$

$$\frac{1}{p_i} + \frac{A(\frac{b}{a})^2}{1+\frac{b}{a}p_i} > 0$$

This will be always satisfied if A> 0, if A is negative, it will still be satisfied

$A\,(b/a)^2 > -\left(\frac{1}{P_i}+\frac{b}{a}\right)$

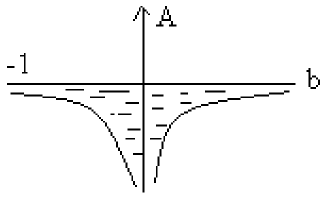
$$\text{i.e. } -\frac{1}{P_i} < A\left(\frac{b}{a}\right)^2 + \frac{b}{a} \quad (4)$$

Now, $\frac{1}{p_i}$ varies from 1 to ∞ , so that (4) will satisfied if

$$A\left(\frac{b}{a}\right)^2 + \frac{b}{a} > -1 \text{ or } A > -\left(\frac{a}{b}\right) - \left(\frac{a}{b}\right)^2 \quad (5)$$

The graph of $A = -\left(\frac{a}{b}\right) - \left(\frac{a}{b}\right)^2$

When b> -1 a>0, a≠ b



Where all points inside the shaded region give permissible values of A, b

Now,

$$\lim_{b\to 0} \sum_{i=1}^{n}(1+\frac{b}{a}p_i) ln\frac{(1+\frac{b}{a}p_i)}{(1+\frac{b}{a}q_i)} = 0 \quad (6)$$

$$\lim_{b\to 0} \frac{a}{b}\sum_{i=1}^{n}(1+\frac{b}{a}p_i) ln\frac{(1+\frac{b}{a})p_i}{1+\frac{b}{a}q_i} \quad (7)$$

$$\sum_{i=1}^{n}(P_i - q_i) = 0 \quad (8)$$

So that for all finite values of A, positive or negative which are independent of b (3) approaches K.L. measures [7] as b→0

Also we can use,

$$D\,(P{:}\,Q) = \sum_{i=1}^{n} P_i ln\frac{P_i}{q_i} - \left(\frac{ac}{b}+\frac{a^2 d}{b^2}\right)\sum_{i=1}^{n}(1+\frac{b}{a}P_i) ln\frac{(1+\frac{b}{a}P_i)}{(1+\frac{b}{a}q_i)} \quad (9)$$

Where c and d are any positive number less than unity.

We consider some cases

When c=1 d=0

Or c=1 d=1

Or c=0 d=1

The measure (9) is again in general not a convex function of $q_1, q_2, \ldots \ldots q_n$

(II) A measure which is a convex function of both P and Q is obtained from Csiszer's [1] measure.

$$\sum_{i=1}^{n} q_i \phi\left(\frac{P_i}{q_i}\right) \quad (10)$$

Where ϕ (.) is a twice differentiable convex function with ϕ (1) =0 by taking

$$\boldsymbol{\phi}(\mathrm{x}) = \mathrm{x}\ ln\ \mathrm{x} - \frac{a}{b}\left(1+\frac{b}{a}x\right) ln \frac{(1+\frac{b}{a}x)}{(1+\frac{b}{a})} \qquad \mathrm{b>0,\ a\text{-}>1} \quad (11)$$

This gives

$$\mathrm{D\ (P{:}\ Q)} = \sum_{i=1}^{n} P_i ln\frac{P_i}{q_i} - \frac{a}{b}\sum_{i=1}^{n}(q_i+\frac{b}{a}P_i) ln \frac{q_i+\frac{b}{a}p_i}{q_i(1+\frac{b}{a})} \quad (12)$$

It can be generalized as

$$\boldsymbol{\phi}(\mathrm{x}) = \mathrm{x}\ ln\ x + A\left(1+\frac{b}{a}x\right) ln \frac{(1+\frac{b}{a}x)}{1+\frac{b}{a}} \quad (13)$$

$$\boldsymbol{\phi}'(\mathrm{x}) = lnx + A\left(1+\frac{b}{a}x\right) ln \frac{(1+\frac{b}{a}x)}{1+\frac{b}{a}} + A\left(\frac{b}{a}\right) + 1$$

$$\boldsymbol{\phi}''(\mathrm{x}) = \frac{1}{x} + \frac{A(\frac{b}{a})^2}{(1+\frac{b}{a}x)}$$

It will be convex if

$$\left.\begin{aligned} &\frac{1}{x} + \frac{A\left(\frac{b}{a}\right)^2}{\left(1+\frac{b}{a}x\right)} > 0 \\ or\ &A\left(\frac{b}{a}\right)^2 + \left(\frac{b}{a}\right) > -1/x \end{aligned}\right\} \quad (14)$$

Now x can vary from 0 to ∞ so that $-\frac{1}{x}$ can vary from - ∞ to 0 so that the condition becomes

$$\mathrm{A}(\frac{b}{a})^2 + \left(\frac{b}{a}\right) > 0 \quad \text{or} \quad \mathrm{A> -\ a/b} \quad (15)$$

Thus, the generalized measure of directed divergence which is a convex function of both P and Q is,

$$\mathrm{D\ (P{:}\ Q)} = \sum_{i=1}^{n} P_i \ln\frac{P_i}{q_i} - A\sum_{i=1}^{n}\left(q_i+\frac{b}{a}p_i\right) \ln \frac{q_i+\frac{b}{a}p_i}{q_i(1+\frac{b}{a})} \quad (16)$$

Where A is any positive number or a negative number $\geq -b/a$

(III) Now consider

$$\boldsymbol{\phi}(\mathrm{x}) = \frac{x^{\propto}-x}{1-\propto} + A\frac{(1+\frac{b}{a}x)^{\propto}-(1+\frac{b}{a}x)}{\propto-1} - A\frac{(1+\frac{b}{a})^{\propto}-(1+\frac{b}{a})}{\propto-1} \quad (17)$$

$$\boldsymbol{\phi}''(\mathrm{x}) = \propto x^{\propto-2} + A\propto\left(1-\frac{b}{a}x\right)^{\propto-2}\left(\frac{b}{a}\right)^2$$

This will be convex if

$$\propto x^{\propto-2} + A\propto(1+\frac{b}{a}x)^{\propto-2} \geq 1 \quad (18)$$

If A is positive, this is always satisfied

$$[\frac{x}{1+\frac{b}{a}x}]^{\propto-2} \quad \geq -A$$

If A = -B this gives,

$$\left.\begin{aligned} &\left[\frac{x}{1+\frac{b}{a}x}\right]^{\propto-2} \geq B \\ or\ &\left[\frac{1+\frac{b}{a}x}{x}\right]^{2-\propto} \geq B \end{aligned}\right] \quad (19)$$

As x goes from 0 to∞,

$\frac{x}{1+\frac{b}{a}x}$ Goes from 0 to b/a

If ∝> 2, thus requires B ≤ 0 or B=0

$$\text{If } \propto = 2,\ B \leq 1 \quad (20)$$

If ∝< 2 $(\frac{1+\frac{b}{a}x}{x})$ can vary from b to ∞ a→1

Expression (19) gives

$$\mathrm{B} \leq (b)^{2-\propto} \quad (21)$$

Also

$$\mathrm{D\ (P{:}\ Q)} = \frac{1}{\propto-1}[(\sum_{i=1}^{n} p^{\propto}q^{1-\propto} - 1 + A[\left(q_i+\frac{b}{a}P_i\right)^{\propto}q_i^{1-\propto} - (q_i+ap_i)\Big] - A(1+a)^{\propto} + A(1+a)] \quad (22)$$

Gives a valid measure of directed divergence for all non-negative values of A.

Also,

$$\mathrm{D\ (P{:}\ Q)} = \frac{1}{\propto-1}[\sum_{i=1}^{n} p^{\propto}q^{1-\propto} - 1 - B[(q_i+ap_i)^{\propto}(q_i)^{1-\propto} - (q_i+ap_i)] + B(1+a)^{\propto} - B(1+a)] \quad (23)$$

Gives a valid measure of directed divergence if B≤ $b^{2-\propto}$ when 0≤ $b \leq 2$ and a→1

B=0 when ∝> 2.

2. Conclusion

Directed Divergence opens a gate way to machine learning which is the future study of Information Theory .

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