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Generalized Fractional Calculus: Saigo Operators on Bessel-Mittag-Leffler Function Products with Quantum Applications

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Abstract: ***In the present paper we have established the generalized Saigo's fractional derivative operators of the product of generalized multi-index Bessel function and multi-index Mittag Leffler function. Further, the Riemann-Liouville fractional derivative operator of given functions are obtained.***

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Definitions

1. Generalized Multi-Index Mittag Leffler Function

For $A_j, B_j, \lambda, \rho \in \mathbb{C}$, the generalized multi-index Mittag Leffler function is defined by Saxena and Nishimoto [13] in the following summation form

$$E^{\lambda,\rho}_{(A_j,B_j)_m}(x) = \sum_{k=0}^{\infty} \frac{(\lambda)_{\rho k}}{\prod_{j=1}^{m} \Gamma(A_j\, k + B_j)} \frac{x^k}{k!}; \quad (m \in \mathbb{N}) \quad (1.1)$$

where $\mathcal{R}e(B_j) > 0$ and $\sum_{j=1}^{m} \mathcal{R}e(A_j) > max\,\{\mathcal{R}e(\rho) - 1; 0\}$.

For $m = 1$ the generalized multi-index Mittag Leffler function (1.1) reduce into the generalized Mittag-Leffler function given by Shukla and Prajapati [16] and defined as

$$E^{\lambda,\rho}_{A,B}(x) = \sum_{k=0}^{\infty} \frac{(\lambda)_{\rho k}}{\Gamma(Ak + B)} \frac{x^k}{k!}, \quad (1.2)$$

where $A, B, \lambda \in \mathbb{C}$; $\mathcal{R}e(A) > 0, \mathcal{R}e(B) > 0, \mathcal{R}e(\lambda) > 0$ and $\rho \in (0, 1) \cup \mathbb{N}$

For $m = 1$ and $\rho = 1$, the generalized multi-index Mittag Leffler function (1.1) reduce into the generalized Mittag-Leffler function given by Prabhakar [10] defined as

$$E^{\lambda}_{A,B}(x) = \sum_{k=0}^{\infty} \frac{(\lambda)_k}{\Gamma(Ak + B)} \frac{x^k}{k!}, \quad (1.3)$$

where $A, B, \lambda \in \mathbb{C}$; $\mathcal{R}e(A) > 0$, $\mathcal{R}e(B) > 0$, $\mathcal{R}e(\lambda) > 0, x \in \mathbb{C}$ and $(\lambda)_k$ is the well known Pochhammer symbol.

2. Generalized Multi-Index Bessel Function

For $A_j, B_j, \lambda, \in \mathbb{C}, \mu > 0, \mathcal{R}e(\lambda) > 0$ the generalized multi-index Bessel function is defined by Choi and Agarwal [1] in the following summation form:

$$J^{(A_j)_{m,\lambda}}_{(B_j)_{m,\mu}}(x) = \sum_{l=0}^{\infty} \frac{(\lambda)_{\mu l}}{\prod_{j=1}^{m} \Gamma(A_j\, l + B_j + 1)} \frac{(-x)^l}{l!}; \quad (m \in \mathbb{N}) \quad (2.1)$$

where $\mathcal{R}e(B_j) > -1$ and $\sum_{j=1}^{m} \mathcal{R}e(A_j) > max\,\{0; \mathcal{R}e(\mu) - 1\}$

3. Generalized Fractional Derivative Operators

For $\mu = [\mathrm{Re}(\gamma) + 1]$ and $x > 0$, where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \in \mathbb{C}$ with $\mathrm{Re}(\gamma) > 0$, the generalized fractional derivative operators are defined [12, 14] as follows:

$$D^{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma}_{0+} f(x) = (I^{-\alpha_2,-\alpha_1,-\beta_2,-\beta_1,-\gamma}_{0+} f)(x); \quad \mathrm{Re}(\gamma) > 0$$

$$= \left(\frac{d}{dx}\right)^{\mu} (I^{-\alpha_2,-\alpha_1,-\beta_2+\mu,-\beta_1,-\gamma+\mu}_{0+} f)(x). \quad (3.1)$$

$$D^{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma}_{0-} f(x) = (I^{-\alpha_2,-\alpha_1,-\beta_2,-\beta_1,-\gamma}_{0-} f)(x); \quad \mathrm{Re}(\gamma) > 0$$

$$= \left(-\frac{d}{dx}\right)^{\mu} (I^{-\alpha_2,-\alpha_1,-\beta_2,-\beta_1+\mu,-\gamma+\mu}_{0-} f)(x). \quad (3.2)$$

Where generalized fractional integral operators are defined as

$$(I^{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma}_{0+} f)(x)$$

$$= \frac{x^{-\alpha_1}}{\Gamma(\gamma)} \int_0^x \frac{(x-t)^{\gamma-1}}{t^{\alpha_2}} F_3\left(\alpha_1, \alpha_2, \beta_1, \beta_2; \gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)dt.$$

and

$$(I_{0-}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma} f)(x)$$

$$= \frac{x^{-\alpha_2}}{\Gamma(\gamma)} \int_x^\infty \frac{(t-x)^{\gamma-1}}{t^{\alpha_1}} F_3\left(\alpha_1, \alpha_2, \beta_1, \beta_2; \gamma; 1-\frac{x}{t}, 1-\frac{t}{x},\right) f(t)dt .$$

The following image formula for a power function under the generalized fractional integral operators is given [12,15] as follows:

$$(I_{0+}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma} t^{\varepsilon-1})(x)$$

$$= \Gamma\begin{bmatrix} \varepsilon, & \varepsilon+\gamma-\alpha_1-\alpha_2-\beta_1, & \varepsilon-\alpha_2+\beta_2 \\ \varepsilon+\beta_2, & \varepsilon+\gamma-\alpha_1-\alpha_2, & \varepsilon+\gamma-\alpha_2-\beta_1 \end{bmatrix} x^{\varepsilon-\alpha_1-\alpha_2+\gamma-1}, \quad (3.3)$$

where $\mathrm{Re}(\varepsilon) > max\{\mathrm{Re}(\alpha_1+\alpha_2+\beta_1-\delta),\ \mathrm{Re}(\alpha_2-\beta_2), 0\}$ and

$$(I_{0-}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma} t^{-\varepsilon})(x)$$

$$= \Gamma\begin{bmatrix} \alpha_1+\alpha_2-\gamma+\varepsilon, & \alpha_1+\beta_2-\gamma+\varepsilon, & -\beta_1+\varepsilon \\ \varepsilon, & \alpha_1+\alpha_2+\beta_2-\gamma+\varepsilon, & \alpha_1-\beta_1+\varepsilon \end{bmatrix} x^{-\varepsilon-\alpha_1-\alpha_2+\gamma}, \quad (3.4)$$

Where, $\mathrm{Re}(\varepsilon) < 1 + min\{\mathrm{Re}(-\beta_1),\ \mathrm{Re}(\alpha_1+\beta_2-\gamma),\ \mathrm{Re}(\alpha_1+\alpha_2-\gamma)\}$ and

$$\Gamma\begin{bmatrix} \alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2, & \beta_3 \end{bmatrix} = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)}$$

The generalized fractional derivative operators reduce into the saigo's fractional

derivative operators due to the following relations:

$$(D_{0+}^{0,\alpha_2,\beta_1,\beta_2,\gamma} f)(x) = (D_{0+}^{\gamma,\alpha_2-\gamma,\beta_2-\gamma} f)(x); \mathrm{Re}(\gamma) > 0 \quad (3.5)$$

$$(D_{0-}^{0,\alpha_2,\beta_1,\beta_2,\gamma} f)(x) = (D_{0,\infty}^{\gamma,\alpha_2-\gamma,\beta_2-\gamma} f)(x); \mathrm{Re}(\gamma) > 0 \quad (3.6)$$

$$D_{0+}^{\alpha_1,\beta_1,\gamma} f(x) = (I_{0+}^{-\alpha_1,-\beta_1,\alpha_1+\gamma} f)(x); \mathrm{Re}(\alpha_1) > 0$$

$$= \left(\frac{d}{dx}\right)^\mu (I_{0+}^{-\alpha_1+\mu,-\beta_1-\mu,\alpha_1+\gamma-\mu} f)(x);\ \mu = [\mathrm{Re}(\alpha_1)+1]. \quad (3.7)$$

$$D_{0-}^{\alpha_1,\beta_1,\gamma} f(x) = (I_{0-}^{-\alpha_1,-\beta_1,\alpha_1+\gamma} f)(x); \mathrm{Re}(\alpha_1) > 0$$

$$= \left(-\frac{d}{dx}\right)^\mu (I_{0-}^{-\alpha_1+\mu,-\beta_1-\mu,\alpha_1+\gamma} f)(x); \mu = [\mathrm{Re}(\alpha_1)+1], \quad (3.8)$$

Where saigo's fractional integral operators introduced by saigo's [11] and defined as follows:

$$(I_{0+}^{\alpha_1,\beta_1,\gamma} f)(x)$$

$$= \frac{x^{-\alpha_1-\beta_1}}{\Gamma(\alpha_1)} \int_0^x (x-t)^{\alpha_1-1}\, {}_2F_1\left(\alpha_1+\beta_1, -\gamma;\ \alpha_1; 1-\frac{t}{x}\right) f(t)dt,$$

$$(I_{0-}^{\alpha_1,\beta_1,\gamma} f)(x)$$

$$= \frac{1}{\Gamma(\alpha_1)} \int_x^\infty (t-x)^{\alpha_1-1} t^{-\alpha_1-\beta_1}\, {}_2F_1\left(\alpha_1+\beta_1, -\gamma;\ \alpha_1; 1-\frac{x}{t}\right) f(t)dt.$$

The following image formula for a power function under the saigo's fractional

integral operators is given [12] as follows:

$$(I_{0+}^{\alpha_1,\beta_1,\gamma} t^{\varepsilon-1})(x) = \frac{\Gamma(\varepsilon)\Gamma(\varepsilon+\gamma-\beta_1)}{\Gamma(\varepsilon+\alpha_1+\gamma)\Gamma(\varepsilon-\beta_1)} x^{\varepsilon-\beta_1-1} \quad (3.9)$$

where $\mathrm{Re}(\varepsilon) > max\{0, \mathrm{Re}(\beta_1-\gamma)\}$ and

$$(I_{0-}^{\alpha_1,\beta_1,\gamma} t^{\varepsilon-1})(x) = \frac{\Gamma(\beta_1-\varepsilon+1)\Gamma(\gamma-\varepsilon+1)}{\Gamma(1-\varepsilon)\Gamma(\alpha_1+\beta_1+\gamma-\varepsilon+1)} x^{\varepsilon-\beta_1-1} \quad (3.10)$$

where $\mathrm{Re}(\varepsilon) < 1 + min\{\mathrm{Re}(\beta_1), \mathrm{Re}(\gamma)\}$.

Let $f_1(x) = \sum_{k=0}^\infty C_k x^k$ and $f_2(x) = \sum_{k=0}^\infty D_k x^k$ be two analytic fuctions with their

radii of convergence R_{f_1} and R_{f_2}, respectively. Then their Hadamard product [13,16] is

given by the following power series:

$$f_1 * f_2(x) = f_2 * f_1(x) = \sum_{k=0}^\infty C_k D_k x^k;\ (|x| < R), \quad (3.11)$$

Where $R_c \geq R_{f_1}.R_{f_2}$ is the radius of convergence of the composite series.

4. Main Results

Theorem 1. Let $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \in \mathbb{C}$ be such that $x > 0, \mathrm{Re}(\gamma) > 0$ and the conditions given in (1.1),(2.1) and (3.7) be satisfied. Then the left sided Saigo's derivative of the product of generalized multi-index Bessel function $J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x)$ and multi-index Mittag Leffler function $E_{(A_j,B_j)_m}^{\lambda,\rho}(x)$ is given by

$$[D_{0+}^{\alpha_1,\beta_1,\gamma}\{t^{\delta-1}J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x_1t)\times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2t)\}](x) = x^{\delta+\beta_1-1}\left\{J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x_1)\times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2)\right\}$$

$$\otimes\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma(l+k+\delta)\Gamma(\alpha_1+\beta_1+\gamma+\delta+l+k)}{\Gamma(l+k+\delta+\gamma)\Gamma(\delta+l+k+\beta_1)}x^{l+k}, \qquad (4.1)$$

Where ⊗ stands for convolution product of two functions

Proof. We refer to the left hand side of equation (4.1) by the symbol D_1.

Then making the use of equation (1.1),(2.1)and (3.7) in (4.1), we have

$$D_1 \equiv \left[D_{0+}^{\alpha_1,\beta_1,\gamma}\left\{t^{\delta-1}\sum_{l=0}^{\infty}\frac{(\lambda)_{\mu l}}{\prod_{j=1}^{m}\Gamma(A_j\,l+B_j+1)}\frac{(-x_1t)^l}{l!}\times\sum_{k=0}^{\infty}\frac{(\lambda)_{\rho k}}{\prod_{j=1}^{m}\Gamma(A_j\,k+B_j)}\frac{(x_2t)^k}{k!}\right\}\right](x)$$

After changing the order of summations and derivative operator under the conditions of theorem, we obtain the above as

$$=\sum_{l=0}^{\infty}\frac{(\lambda)_{\mu l}}{\prod_{j=1}^{m}\Gamma(A_j\,l+B_j+1)}\frac{(-x_1)^l}{l!}\times\sum_{k=0}^{\infty}\frac{(\lambda)_{\rho k}}{\prod_{j=1}^{m}\Gamma(A_j\,k+B_j)}\frac{(x_2)^k}{k!}\times[D_{0+}^{\alpha_1,\beta_1,\gamma}(t^{\delta+k+l-1})](x)$$

$$=\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\lambda)_{\mu l}(\lambda)_{\rho k}}{\prod_{j=1}^{m}\Gamma(A_j\,l+B_j+1)\times\prod_{j=1}^{m}\Gamma(A_j\,k+B_j)}\frac{(-x_1)^l}{l!}\frac{(x_2)^k}{k!}$$

$$\times[I_{0+}^{-\alpha_1,-\beta_1,\alpha_1+\gamma}(t^{\delta+k+l-1})](x).$$

Using the image formula for power function under generalized operator (3.9), we get

$$=\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\lambda)_{\mu l}(\lambda)_{\rho k}}{\prod_{j=1}^{m}\Gamma(A_j\,l+B_j+1)\times\prod_{j=1}^{m}\Gamma(A_j\,k+B_j)}$$

$$\times\frac{(-x_1)^l}{l!}\frac{(x_2)^k}{k!}x^{\delta+k+l+\beta_1-1}$$

$$\frac{\Gamma(k+l+\delta)\Gamma(\alpha_1+\beta_1+\gamma+\delta+k+l)}{\Gamma(k+l+\delta+\gamma)\Gamma(\delta+k+l+\beta_1)}$$

Further, applying the definition (1.1) and (2.1) and convolution product on two series, we obtain

$$D_1 \equiv x^{\delta+\beta_1-1}\left\{J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x_1)\times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2)\right\}$$

$$\otimes\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma(l+k+\delta)\Gamma(\alpha_1+\beta_1+\gamma+\delta+l+k)}{\Gamma(l+k+\delta+\gamma)\Gamma(\delta+l+k+\beta_1)}x^{l+k}$$

Where ⊗ stands for convolution product of two functions

Theorem 2. Let α_1, α_2, β_1, β_2, $\gamma\in\mathbb{C}$ be such that $x>0$, $\mathrm{Re}(\gamma)>0$ and the conditions given in (1.1),(2.1) and (3.8) be satisfied. Then the right sided Saigo's derivative of the product of generalized multi-index Bessel function $J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x)$ and multi-index Mittag Leffler function $E_{(A_j,B_j)_m}^{\lambda,\rho}(x)$ is given by

$$[D_{0-}^{\alpha_1,\beta_1,\gamma}\{t^{-\delta}J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x_1t)\times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2t)\}](x)$$

$$= x^{-\delta+\beta_1}\left\{J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x_1)\times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2)\right\}$$

$$\otimes\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma(\delta-\beta_1-l-k)\Gamma(\alpha_1+\gamma+\delta-l-k)}{\Gamma(\delta-l-k)\Gamma(\delta+\gamma-\beta_1-l-k)}x^{l+k} \qquad (4.2)$$

Where ⊗ stands for convolution product of two functions.

Proof: We refer to the left hand side of equation (4.2) by the symbol D_2.

Then making the use of equation (1.1),(2.1)and (3.8) in (4.2), we have

$$D_2 \equiv \left[D_{0-}^{\alpha_1,\beta_1,\gamma}\left\{t^{-\delta}\sum_{l=0}^{\infty}\frac{(\lambda)_{\mu l}}{\prod_{j=1}^{m}\Gamma(A_j\,l+B_j+1)}\frac{(-x_1t)^l}{l!}\times\sum_{k=0}^{\infty}\frac{(\lambda)_{\rho k}}{\prod_{j=1}^{m}\Gamma(A_j\,k+B_j)}\frac{(x_2t)^k}{k!}\right\}\right](x)$$

After changing the order of summations and derivative operator under the conditions of theorem, we obtain the above as

$$= \sum_{l=0}^{\infty} \frac{(\lambda)_{\mu l}}{\prod_{j=1}^{m} \Gamma(A_j\, l + B_j + 1)} \frac{(-x_1)^l}{l!} \times \sum_{k=0}^{\infty} \frac{(\lambda)_{\rho k}}{\prod_{j=1}^{m} \Gamma(A_j\, k + B_j)} \frac{(x_2)^k}{k!}$$

$$\times [D_{0-}^{\alpha_1,\beta_1,\gamma}(t^{-\delta+l+k})](x)$$

$$= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\lambda)_{\mu l}(\lambda)_{\rho k}}{\prod_{j=1}^{m} \Gamma(A_j\, l + B_j + 1) \times \prod_{j=1}^{m} \Gamma(A_j\, k + B_j)} \frac{(-x_1)^l}{l!} \frac{(x_2)^k}{k!}$$

$$\times [I_{0-}^{-\alpha_1,-\beta_1,\alpha_1+\gamma}(t^{-\delta+l+k})](x)$$

Using the image formula for power function under generalized operator (3.10), we get

$$= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\lambda)_{\mu l}(\lambda)_{\rho k}}{\prod_{j=1}^{m} \Gamma(A_j\, l + B_j + 1) \times \prod_{j=1}^{m} \Gamma(A_j\, k + B_j)} \times \frac{(-x_1)^l}{l!} \frac{(x_2)^k}{k!} x^{\beta_1-\delta+l+k}$$

$$\times \frac{\Gamma(\delta - \beta_1 - l - k)\Gamma(\alpha_1 + \gamma + \delta - l - k)}{\Gamma(\delta - l - k)\Gamma(\delta + \gamma - \beta_1 - l - k)}$$

Further, applying the definition (1.1) and (2.1) and convolution product on two series, we obtain

$$D_2 \equiv x^{-\delta+\beta_1} \left\{ J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x_1) \times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2) \right\}$$

$$\otimes \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma(\delta - \beta_1 - l - k)\Gamma(\alpha_1 + \gamma + \delta - l - k)}{\Gamma(\delta - l - k)\Gamma(\delta + \gamma - \beta_1 - l - k)} x^{l+k}$$

Where ⊗ stands for convolution product of two functions.

5. Special Cases

In this section, we will derive the following new composite formulas with the help of the main results

Corollary 1. Let the conditions of Theorem 1 be satisfied and $\alpha_1 = 0$, $\beta_1 = 1$ then the Theorem 1 reduced in the following form:

$$[D_{0+}^{0,1,\gamma}\{t^{\delta-1} J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x_1 t) \times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2 t)\}](x)$$

$$= x^{\delta} \left\{ J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x_1) \times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2) \right\}$$

$$\otimes \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{(l + k + \delta + \gamma)}{(l + k + \delta)} x^{l+k} \qquad (5.1)$$

Where ⊗ stands for convolution product of two functions

Corollary 2. Let the conditions of Theorem 2 be satisfied and $\alpha_1 = 0$, $\gamma = 1$ then the Theorem 2 reduced in the following form:

$$[D_{0-}^{0,\beta_1,1}\{t^{-\delta} J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x_1 t) \times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2 t)\}](x)$$

$$= x^{-\delta+\beta_1} \left\{ J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x_1) \times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2) \right\}$$

$$\otimes \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\delta - l - k)}{(\delta - \beta_1 - l - k)} x^{l+k} \qquad (5.2)$$

Where ⊗ stands for convolution product of two functions

References

[1] **Choi, J. and Agarwal, P.:** A note on fractional integral operator associated with multiindex Mittag-leffler functions, Filomat, Vol. 30(7) (2016):1931-1939.

[2] **Choi, J. Kumar, D. and Purohit, S. D.:** Integral formulas involving a product of generalized Bessel functions of the first kind, Kyungpook Math. J. Vol.56(1) (2016):131-136.

[3] **Erdelye, A. et al.:** Higher Transcendental Fucntions,Vol. I, McGraw-Hill, New York London (1953).

[4] **Kilbas, A. A. and Sebastian, N.:** Generalized fractional integration of Bessel Function of first kind, Integral Transform Special Function, Vol. 19(2008):869-883.

[5] **Kataria, K. K. and Vellaisamy, P.:** The k-Wright function and Marichev-saigo-Maeda fractional operators, J. Anal. Vol. 23(2015):75-87.

[6] **Lavoie, J. L. and Trottier, G.:** On the sum of certain Appell's series, Ganita 20(1969):43- 46

[7] **Mishra, V. N. Suthar, D. L. and Purohit, S.D.:** Marichev-Saigo-Maeda fractional calculus operators, Srivastava polynomials and generalized mittag-Leffler function, Cogent mathematics 4(2017)

[8] **Oberhettinger, F**.: Tables of Mellin Transforms, Springer, New York(1947).

[9] **Pohlen, T.:** The Hadamard product and universal power series, Ph.D. Thesis, Universitat Trier, Trier, Germany(2009).

[10] **Prabhakar, T. R.:** A singular integral equation with a generalized Mittag Leffler function in the kernel, Yokohama Math. J.Vol.19(1971):7-15.

[11] **Saigo, M.:** A remark on integral operators involving the Gauss hypergeometric functions. Math. Rep. College General Ed Kyushu univ., Vol.11(1978):135-143.

[12] **Saigo, M. and Maeda, N.:** More generalization of fractional calculus, transform Method and Special

Functions., Bulgarian Acadmey of Sciences, Sofia Vol. 96(1998):386-400.

[13] **Saxena, R.K. and Nishimoto, K.:** N-Fractional calculus of generalized Mittag-Leffler functions, J. Frac. Calc., Vol. 37(2010):43-52.

[14] **Saxena, R.K. and Saigo, M.:** Generalized fractional calculus of the H-function associated with the Appell function F_3, Journal of Fractional calculus, Vol. 19(2001): 89-104.

[15] **Saxena, R.K. and Parmar, R.K.:** Fractional intergration and differention of the generalized Mathieu series, axioms 6(2017):18.

[16] **Shukla, A. K. and Prajapati, J.C.:** On a generalization of Mittag Leffler function and its properties, J. Math. Ann. Appl.,Vol. 336(2)(2007): 797-811.

[17] **Srivastava, H.M.:** A contour integral involving Fox's H-function, India J. Math. Vol.14(1972):1-6

[18] **Wright, E. M.:** The asymptotic expansion of the generalized hypergeometric function ll, Proc. London Math. Soc. Vol. 46(2) (1940):389-408.