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Prcing Maximum Value Options under the Mixed Fractional Brownian Motion with Jumps

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Abstract: ***In this paper, a semi-closed analytical formula for the values of European call options on the maximum of two-asset options under the Mixed Fractional Brownian Motion model with Jumps (JMFBM) are derived by measure transform and equivalent martingale.***

Keywords: Extremum options, Mixed fractional Brownian motion, Jump diffusion, Equivalent martingale.

1. Introduction

Options are very popular financial derivatives and have always played an important role in the financial markets. In 1973, Black and Scholes [1] proposed a famous option pricing model, which laid the foundation for later scholars to study the application of options in financial markets. In response to the development of the financial market, financial institutions have developed a variety of exotic options, among which multi-asset options have attracted the attention of many scholars and investors. The rainbow option is a type of multi-asset option, first proposed by Margrabe [2] in 1978. Its purpose is to maximize returns among multiple risk assets. A few years later, Stulz [3] derived a semi-closed analytical formula for European call and put options under the two-asset Black-Scholes (in short, B-S) model, which is based on the minimum and maximum values of the two underlying. Rubinstein [4] derives a pricing formula for rainbow options under the assumption of risk-neutral, which relies on the maximum or minimum value of the underlying asset price.

The option pricing problem is usually studied on the classical B-S model [1], where the stock price is described by a Geometric Brownian motion (in short, GBM). However, B. B. Mandelbrot and J. W. Van Ness [5] observed long-range dependence of stock returns and gave the definition of Fractional Brownian motion (in short, FBM). Many subsequent scholars have applied Fractional Brownian motion to simulate stock price volatility. jork and Hult [6] and Kuznetsov [7] found that this is not reasonable because it has the possibility of arbitrage. To address the problem of arbitrage, Cheridito [8] argued that it is more reasonable to use the Mixed Fractional Brownian motion (in short, MFBM) to model the volatility of financial assets. The mixed Fractional Brownian motion is a Gaussian process that is a linear combination of a Brownian motion and a Fractional Brownian motion with Hurst exponent H > 1/2. Cheridito [9] proved that there is no possibility of arbitrage in the market when Hurst parameter $H \in (3/4, 1)$ in MFBM model.

Financial markets are susceptible to a variety of unexpected events, and the Mixed Fractional Brownian Motion model with jumps is a versatile and expressive model that combines the jump diffusion process with the characteristics of Mixed Fractional Brownian Motion, which can not only simulate large jumps and small fluctuations, but also capture the long-term memorability and non-Gaussian behavior of the market. Therefore, in order to be closer to the actual situation, this paper introduces the jump process to describe the dynamic process of the underlying asset based on the Mixed Fractional Brownian motion, which is referred to as the JMFBM model. And based on the JMFBM model, a semi-closed analytical formula for the two-asset European maximum call option is derived.

2. The two-asset JMFBM model

Basic Setting of the Pricing Model. $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\}$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t\}$ for $0 \leq t \leq T$, and P a risk-neutral measure. Under the risk-neutral P measure, the return dynamics under the one-asset JMFBM model are given by the following stochastic differential equation (SDE):

$$\frac{dS_t}{S_{t^-}} = (r - \lambda\kappa)dt + \alpha dW(t) + \beta dW^H(t) + \left(e^J - 1\right)dN_t \qquad (1)$$

where $S = \{S_t, t \geq 0\}$ denotes the stock price process, r the risk-free interest rate, $W = \{W(t),\ t \geq 0\}$ is the standard Brownian motion and $W^H = \{W^H(t),\ t \geq 0\}$ is an independent standard Fractional Brownian motion with Hurst index $H \in (3/4, 1)$ and α, β are some real constants not both zero, $N = \{N_t, t > 0\}$ is the Poisson arrival process with jump intensity λ and J the random jump size of the log-returns with expected relative jump size κ. Applying fractional Itô's formula to (2.1) yields the exact solution:

$$S_t = S_0 \exp\left((r - \frac{1}{2}\alpha^2 - \lambda\kappa)t - \frac{1}{2}\beta^2 t^{2H} + \alpha W(t) + \beta W^H(t) + \sum_{k=1}^{N_t} J_k \right) \quad (2)$$

where J_k, k = 1, 2, 3,... are independent jump sizes that have an identical distribution to the random jump size J.

Correspondingly, under the two-asset JMFBM model, the asset prices are given by:

$$\begin{aligned} S_t^{(1)} &= S_0^{(1)} \exp\left((r - \frac{1}{2}\alpha_1^2 - \lambda\kappa_1)t - \frac{1}{2}\beta_1^2 t^{2H} + \alpha_1 W_1(t) + \beta_1 W_1^H(t) + \sum_{k=1}^{N_t} J_k^{(1)} \right) \\ S_t^{(2)} &= S_0^{(2)} \exp\left((r - \frac{1}{2}\alpha_2^2 - \lambda\kappa_2)t - \frac{1}{2}\beta_2^2 t^{2H} + \alpha_2 W_2(t) + \beta_2 W_2^H(t) + \sum_{k=1}^{N_t} J_k^{(2)} \right) \end{aligned} \quad (3)$$

Where $\mathbf{W}_t$, $\mathbf{W}_t^H$, $\mathbf{J}$, and N_t are independent of each other.

◎ $\mathbf{W_t} = [W_1(t)\ \ W_2(t)]^\top$ consists of two correlated standard Brownian motions, having correlation ρ with $|\rho| \le 1$, adapted to the filtration.

◎ $\mathbf{W_t^H} = [W_1^H(t)\ \ W_2^H(t)]^\top$ consists of two correlated standard Fractional Brownian motions, having correlation ρ_H with $|\rho_H| \le 1$, adapted to the filtration;

◎ $\mathbf{J} = [J^{(1)}\ \ J^{(2)}]^\top$, $J_k^{(i)}$ (k = 1, 2, 3, ...) are independent jump sizes that have an identical distribution to the random jump size $J^{(i)}$, $i = 1, 2$ and denote by $N(\varepsilon, \xi^2)$ the univariate normal distribution with mean ε and standard deviation ξ. $J^{(i)} \sim N(\gamma_i, \delta_i^2)$, $i = 1, 2$ .The kth junps $J^{(i)}(i = 1, 2)$ occur together, driven by the same Poisson arrival process N_t with intensity λ ,and are correlated with correlation ρ_J, $|\rho_J| \le 1$.

κ_i is the expected relative jump size of asset i:

$$\kappa_i = E_P[e^{J^{(i)}} - 1] = e^{\gamma_i + \frac{1}{2}\delta_i^2}$$

3. Pricing Formula

With the explicit solution of the stock price $S_t^{(i)}$ in hand, in this section the pricing formula for European call options on the maximum of two-asset options under the JMFBM model can be derived.

Definition 1. A European call-on-the-max option gives its purchaser the right to buy the most expensive asset at the strike price K at expiration T. Its payoff is given by:

$$\begin{aligned} V_{\max-on-call} &= \left[\max(S_T^{(1)}, S_T^{(2)}) - K\right]^+ \\ &= (S_T^{(1)} - K)1_{\mathcal{A}1} + (S_T^{(2)} - K)1_{\mathcal{A}2} \\ &= S_T^{(1)} 1_{\mathcal{A}1} + S_T^{(2)} 1_{\mathcal{A}2} - K1_{\mathcal{A}3} \end{aligned} \tag{4}$$

where $1_{\mathcal{A}}$ denotes the indicator function of an event $\mathcal{A}$.

$$\begin{aligned} \mathcal{A}_1 &= \{S_T^{(1)} \ge S_T^{(2)} \text{ and } S_T^{(1)} > K\} \\ \mathcal{A}_2 &= \{S_T^{(2)} \ge S_T^{(1)} \text{ and } S_T^{(2)} > K\} \\ \mathcal{A}_3 &= \{S_T^{(1)} > K \quad \text{or} \quad S_T^{(2)} > K\} \end{aligned} \tag{5}$$

Accordingly, the value of a European call-on-the-max option under JMFBM model at time zero is given by:

$$\begin{aligned} & C_{\max}(S_T^{(1)}, S_T^{(1)}, T; K) \\ &= e^{-rT} E_P[S_T^{(1)} 1_{\mathcal{A}1} + S_T^{(2)} 1_{\mathcal{A}2} - K1_{\mathcal{A}3} \,|\, \mathcal{F}_0] \\ &= S_T^{(1)} E_P[\exp\{(-\frac{1}{2}\alpha_1^2 - \lambda\kappa_1)T - \frac{1}{2}\beta_1^2 T^{2H} \\ &+ \alpha_1 W_1(T) + \beta_1 W_1^H(T) + \sum_{k=1}^{N_T} J_k^{(1)}\} 1_{\mathcal{A}1} \,|\, \mathcal{F}_0] \\ &+ S_T^{(2)} E_P[\exp\{-\frac{1}{2}\alpha_2^2 - \lambda\kappa_2)T - \frac{1}{2}\beta_2^2 T^{2H} \\ &+ \alpha_2 W_2(T) + \beta_2 W_2^H(T) + \sum_{k=1}^{N_T} J_k^{(2)}\} 1_{\mathcal{A}2} \,|\, \mathcal{F}_0] \\ &- e^{-rT} K E_P[1_{\mathcal{A}3} \,|\, \mathcal{F}_0] \\ &= I_1 + I_2 - I_3 \end{aligned} \tag{6}$$

Next, calculate I_1, I_2, I_3.

Considering the equivalent martingale measure Q_1 of P, where the measurement transformations are performed with reference to part 3 of [10] and [11].

Under the r the probability measure space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, P\}$, over the time interval [0, T].

$$\begin{aligned} L_t &= \left.\frac{dQ_1}{dP}\right|_t \\ &= \exp\left[-(\Sigma^{-1}\boldsymbol{\theta})^\top \mathbf{W}_t - \frac{1}{2}\boldsymbol{\theta}^\top \Sigma^{-1}\boldsymbol{\theta} t\right] \\ &\times \exp\left[-(\Sigma^{-1}\boldsymbol{\theta}_H)^\top \mathbf{W}_t^H - \frac{1}{2}\boldsymbol{\theta}_H{}^\top \Sigma_H{}^{-1}\boldsymbol{\theta}_H t^{2H}\right] \\ &\times \exp\left[-\lambda\kappa t + \sum_{k=1}^{N_t}(\mathbf{a}^\top \mathbf{J_k} + \nu)\right] \end{aligned} \tag{7}$$

Where $E_P[L_t] = 1$, L_t is a Radon–Nikodým derivative of some equivalent measure Q_1 with respect to P,

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},\ \Sigma_H = \begin{bmatrix} 1 & \rho_H \\ \rho_H & 1 \end{bmatrix},\ \boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix},\ \boldsymbol{\theta}_H = \begin{bmatrix} \theta_{H1} \\ \theta_{H2} \end{bmatrix},\ \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix},$$

$\nu \in \mathbb{R}$, $\kappa = e^{\nu} E_Q[e^{\mathbf{a}^\top \mathbf{J}}] - 1$,

and exist standard Brownian motions $\tilde{W}_1(t)$, $\tilde{W}_2(t)$, Fractional Brownian motions $\tilde{W}_1^H(t)$, $\tilde{W}_2^H(t)$ under Q_1 such that

$$d\tilde{W}_1(t) = \theta_1 dt + dW_1(t),\ d\tilde{W}_2(t) = \theta_2 dt + dW_2(t),$$

$$d\tilde{W}_1^H(t) = \theta_1 dt + dW_1^H(t),\ d\tilde{W}_2^H(t) = \theta_2 dt + dW_2^H(t).$$

$d\tilde{W}_1(t)d\tilde{W}_2(t) = \rho dt$, $d\tilde{W}_2^H(t)d\tilde{W}_2^H(t) = \rho_H dt$.

Under Q_1 , the compound Poisson process $\sum_{k=1}^{N_t} \mathbf{J}_k$ has an intensity rate: $\tilde{\lambda} = \lambda(1+\kappa)$.

First, consider the term of I_1 :

$$E_P[\exp\{-\frac{1}{2}\alpha_1^2 - \lambda\kappa_1)T - \frac{1}{2}\beta_1^2 T^{2H} + \alpha_1 W_1(T) + \beta_1 W_1^H(T) + \sum_{k=1}^{N_T} J_k^{(1)}\}1_{\mathcal{A}1} \mid \mathcal{F}_0\} \tag{8}$$

Set

$$L_t^{(1)} = \exp\{(-\frac{1}{2}\alpha_1^2 - \lambda\kappa_1)t - \frac{1}{2}\beta_1^2 t^{2H} + \alpha_1 W_1(t) + \beta_1 W_1^H(t) + \sum_{k=1}^{N_t} J_k^{(1)}\} \tag{9}$$

Then $L_t^{(1)}$ is formally identical to (7). For the first Brownian motions part in (7) there holds

$$\begin{aligned}&\exp\left[-(\Sigma^{-1}\boldsymbol{\theta})^\top \mathbf{W}_t - \frac{1}{2}\boldsymbol{\theta}^\top \Sigma^{-1}\boldsymbol{\theta} t\right] \\ &= \exp\left[\frac{\theta_2\rho - \theta_1}{1-\rho^2} W_1(t) + \frac{\theta_1\rho - \theta_2}{1-\rho^2} W_2(t) - \frac{1}{2} t \frac{\theta_1^2 - 2\rho\theta_1\theta_2 + \theta_2^2}{1-\rho^2}\right] \\ &= \exp[-\frac{1}{2}\alpha_1^2 t - \alpha_1 W_1(t)]\end{aligned} \tag{10}$$

Comparing this to (9),

$$\frac{\theta_2\rho - \theta_1}{1-\rho^2} = \alpha_1, \quad \frac{\theta_1\rho - \theta_2}{1-\rho^2} = 0, \quad \frac{\theta_1^2 - 2\rho\theta_1\theta_2 + \theta_2^2}{1-\rho^2} = \alpha_1^2$$

Which is satisfied for

$$\theta_1 = -\alpha_1, \ \theta_2 = -\rho\alpha_1$$

Similarly, for the Fractional Brownian motions part in (7) there holds

$$\begin{aligned}&\exp\left[-(\Sigma_H^{-1}\boldsymbol{\theta}_H)^\top \mathbf{W}_t^H - \frac{1}{2}\boldsymbol{\theta}_H^\top \Sigma^{-1}\boldsymbol{\theta}_H t^{2H}\right] \\ &= \exp\left[\frac{\theta_{H2}\rho_H - \theta_{H1}}{1-\rho_H^2} W_1^H(t) + \frac{\theta_{H1}\rho_H - \theta_{H2}}{1-\rho_H^2} W_2^H(t) - \frac{1}{2} t^{2H} \frac{\theta_{H1}^2 - 2\rho_H\theta_{H1}\theta_{H2} + \theta_{2H}^2}{1-\rho_H^2}\right] \\ &= \exp[-\frac{1}{2}\beta_1^2 t^{2H} + \beta_1 W_1^H(t)]\end{aligned} \tag{11}$$

Which is satisfied for

$$\theta_{H1} = -\beta_1, \ \theta_{H2} = -\rho_H\beta_1$$

Similarly, for the Jump process part in (3.3) there holds

$$\exp\left[-\lambda\kappa t + \sum_{k=1}^{N_t}(\mathbf{a}^\top \mathbf{J}_k + \nu)\right] = \exp[\sum_{k=1}^{N_t} J_k^{(1)}] \tag{12}$$

Which is satisfied for $\nu = 0$, $\mathbf{a} = [1 \ \ 0]^\top$, and hence $\kappa = \kappa_1$. Under Q_1 ,The moment generating function for the kth jump sizes $\mathbf{J}_k = [J_k^{(1)} \ \ J_k^{(2)}]^\top$ is defined by:

$$\begin{aligned}M_{Q_1,\mathbf{J}}(\mathbf{u}) &= \frac{M_{P,\mathbf{J}}(\mathbf{u}+\mathbf{a})}{M_{P,\mathbf{J}}(\mathbf{a})} \\ &= \exp\{(\gamma_1 + \delta_1^2)u_1 + (\gamma_2 + \hat{\rho}\delta_1\delta_2)u_2 + \frac{1}{2}(u_1^2\delta_1^2 + 2\hat{\rho}\delta_1\delta_2 u_1 u_2 + u_2^2\delta_2^2)\}\end{aligned} \tag{13}$$

Hence, $\mathbf{J}_k$ is bivariate normally distributed under Q_1 with

$$\tilde{J}_k^{(1)} \sim N(\gamma_1 + \delta_1^2, \ \delta_1^2), \ \tilde{J}_k^{(2)} \sim N(\gamma_2 + \rho_J\delta_1\delta_2, \ \delta_2^2) \tag{14}$$

and covariance matrix

$$\sum_J = \begin{bmatrix} 1 & \rho_J \\ \rho_J & 1 \end{bmatrix} \tag{15}$$

Consequently

$$\begin{aligned}&E_P[\exp\{-\frac{1}{2}\alpha_1^2 - \lambda\kappa_1)T - \frac{1}{2}\beta_1^2 T^{2H} + \alpha_1 W_1(T) + \beta_1 W_1^H(T) + \sum_{k=1}^{N_T} J_k^{(1)}\}1_{\mathcal{A}1} \mid \mathcal{F}_0\} \\ &= E_{Q_1}[1_{\mathcal{A}1} \mid \mathcal{F}_0] \\ &= Q_1[\mathcal{A}_1 \mid \mathcal{F}_0] \\ &= \sum_{n=0}^{\infty} e^{-\lambda_1 T}\frac{(\lambda_1 T)^n}{n!} Q_1(\mathcal{A}_1 \mid N_T = n, \ \mathcal{F}_0)\end{aligned} \tag{16}$$

Next calculate the probability $Q_1(\mathcal{A}_1 \mid N_T = n, \ \mathcal{F}_0)$.

$$\{S_T^{(1)} \geq S_T^{(2)} \mid N_T = n, \ \mathcal{F}_0\}$$

$$\Leftrightarrow$$

$$\{\ln\frac{S_0^{(1)}}{S_0^{(2)}} + [\frac{1}{2}\alpha_2^2 - \frac{1}{2}\alpha_1^2 + \lambda(\kappa_1 - \kappa_2)]T + (\frac{1}{2}\beta_2^2 - \frac{1}{2}\beta_1^2)T^{2H}\}$$

$$\geq [\alpha_2 W_2(T) - \alpha_1 W_1(T)] + [\beta_2 W_2^H(T) - \beta_1 W_1^H(T)] + \sum_{k=1}^{N_T}[J_k^{(2)} - J_k^{(1)}]\}$$

$$\Leftrightarrow$$

$$\{\ln\frac{S_0^{(1)}}{S_0^{(2)}} + [\frac{1}{2}\alpha_2^2 - \frac{1}{2}\alpha_1^2 - \rho\alpha_1\alpha_2 + \alpha_1^2 + \lambda(\kappa_1 - \kappa_2)]T$$

$$+(\frac{1}{2}\beta_2^2 - \frac{1}{2}\beta_1^2 - \rho_H\beta_1\beta_2 + \alpha_1^2)T^{2H}$$

$$\geq [\alpha_2\tilde{W}_2(T) - \alpha_1\tilde{W}_1(T)] + [\beta_2\tilde{W}_2^H(T) - \beta_1\tilde{W}_1^H(T)] + \sum_{k=1}^{N_T}[\tilde{J}_k^{(2)} - \tilde{J}_k^{(1)}]\}$$

Let $N_T = n$,

$$\begin{aligned}X_{11} &= \ln\frac{S_0^{(1)}}{S_0^{(2)}} + [\frac{1}{2}\alpha_2^2 + \frac{1}{2}\alpha_1^2 - \rho\alpha_1\alpha_2 + \lambda(\kappa_1 - \kappa_2)]T \\ &+(\frac{1}{2}\beta_2^2 + \frac{1}{2}\beta_1^2 - \rho_H\beta_1\beta_2)T^{2H}\end{aligned} \tag{17}$$

$$Y_{11} = [\alpha_2 \tilde{W}_2(T) - \alpha_1 \tilde{W}_1(T)] + [\beta_2 \tilde{W}_2^H(T) - \beta_1 \tilde{W}_1^H(T)] + \sum_{k=1}^{N_T} [\tilde{J}_k^{(2)} - \tilde{J}_k^{(1)}] \tag{18}$$

Where under Q_1

$$Y_{11} \sim N(\mu_{11}, \sigma_{11}^2) \tag{19}$$

calculate mean and variance

$$\mu_{11} = n(\gamma_2 - \gamma_1 + \rho_J \delta_1 \delta_2 - \delta_1^2). \tag{20}$$

$$\sigma_{11}^2 = (\alpha_2^2 + \alpha_1^2 - 2\rho\alpha_1\alpha_2)T + (\beta_2^2 + \beta_1^2 - 2\rho_H \beta_1 \beta_2) T^{2H} + n(\delta_1^2 + \delta_2^2 - 2\rho_J \delta_1 \delta_2) \tag{21}$$

Let

$$d_{11} = \frac{X_{11} - \mu_{11}}{\sigma_{11}} \tag{22}$$

$$Z_{11} = \frac{Y_{11} - \mu_{11}}{\sigma_{11}} \sim N(0,1) \tag{23}$$

Thus

$$\{S_T^{(1)} \geq S_T^{(2)} \mid N_T = n, \mathcal{F}_0\} \Leftrightarrow \{X_{11} \geq Y_{11}\} \Leftrightarrow \{d_{11} \geq Z_{11}\}$$

Similarly,

$$\{S_T^{(1)} > K \mid N_T = n, \mathcal{F}_0\} \Leftrightarrow \{X_{12} \geq Y_{12}\} \Leftrightarrow \{d_{12} \geq Z_{12}\}$$

Where

$$X_{12} = \ln \frac{S_0^{(1)}}{K} + (r + \frac{1}{2}\alpha_1^2 - \lambda\kappa_1)T + \frac{1}{2}\beta_1^2 T^{2H} \tag{24}$$

$$Y_{12} = -\alpha_1 \tilde{W}_1(T) - \beta_1 \tilde{W}_1^H(T) - \sum_{k=1}^{N_T} \tilde{J}_k^{(1)}, \ Y_{12} \sim N(\mu_{12}, \sigma_{12}^2) \tag{25}$$

calculate mean and variance

$$\mu_{12} = -n(\gamma_1 + \delta_1^2) \tag{26}$$

$$\sigma_{11}^2 = \alpha_1^2 T + \beta_1^2 T^{2H} + n\delta_1^2 \tag{27}$$

Let

$$d_{12} = \frac{X_{12} - \mu_{12}}{\sigma_{12}} \tag{28}$$

$$Z_{12} = \frac{Y_{12} - \mu_{12}}{\sigma_{12}} \sim N(0,1) \tag{29}$$

Under Q_1 the Z_{11} and Z_{12} are correlated standard normal random variables, with correlation coefficient

$$\tilde{\rho}_1 = \frac{E_{Q_1}[Y_{11} Y_{12}] - \mu_{11}\mu_{12}}{\sigma_{11}\sigma_{12}} \tag{30}$$

Where

$$E_{Q_1}[Y_{11} Y_{12}] = (\alpha_1^2 - \rho\alpha_1\alpha_2)T + (\beta_1^2 - \rho_H \beta_1 \beta_2) T^{2H} - n^2[\delta_1^2 + \rho_J \delta_1 \delta_2 + (\gamma_1 + \delta_1^2)(\gamma_2 + \rho_J \delta_1 \delta_2)] \tag{31}$$

and covariance matrix

$$\Sigma_1 = \begin{bmatrix} 1 & \tilde{\rho}_1 \\ \tilde{\rho}_1 & 1 \end{bmatrix} \tag{32}$$

Let $\Phi(x, y, \Sigma)$ denote the bivariate normal cumulative distribution function (cdf), evaluated at (x, y), with mean $[0 \ \ 0]^\top$ and covariance matrix Σ.

Thus

$$\begin{aligned} & Q_1(\mathcal{A}_1 \mid \mathcal{F}_0) \\ &= \sum_{n=0}^{\infty} e^{-\lambda_1 T} \frac{(\lambda_1 T)^n}{n!} Q_1(\mathcal{A}_1 \mid N_T = n, \mathcal{F}_0) \\ &= \sum_{n=0}^{\infty} e^{-\lambda_1 T} \frac{(\lambda_1 T)^n}{n!} Q_1(Z_{11} \leq d_{11}, \ Z_{12} < d_{12}) \\ &= \sum_{n=0}^{\infty} e^{-\lambda_1 T} \frac{(\lambda_1 T)^n}{n!} \Phi(d_{11}, \ d_{12}, \Sigma_1) \end{aligned} \tag{33}$$

get

$$I_1 = S_0^{(1)} \sum_{n=0}^{\infty} e^{-\lambda_1 T} \frac{(\lambda_1 T)^n}{n!} \Phi(d_{11}, \ d_{12}, \Sigma_1) \tag{34}$$

Next, calculate I_1. By symmetry:

$$I_2 = S_0^{(2)} \sum_{n=0}^{\infty} e^{-\lambda_2 T} \frac{(\lambda_2 T)^n}{n!} \Phi(d_{21}, \ d_{22}, \Sigma_2) \tag{35}$$

Where

$$X_{21} = \ln \frac{S_0^{(2)}}{S_0^{(1)}} + [\frac{1}{2}\alpha_1^2 + \frac{1}{2}\alpha_2^2 - \rho\alpha_1\alpha_2 + \lambda(\kappa_2 - \kappa_1)]T + (\frac{1}{2}\beta_1^2 + \frac{1}{2}\beta_2^2 - \rho_H \beta_1 \beta_2) T^{2H} \tag{36}$$

$$Y_{21} = [\alpha_1 \tilde{W}_1(T) - \alpha_2 \tilde{W}_2(T)] + [\beta_1 \tilde{W}_1^H(T) - \beta_2 \tilde{W}_2^H(T)] + \sum_{k=1}^{N_T} [\tilde{J}_k^{(1)} - \tilde{J}_k^{(2)}] \tag{37}$$

$$Y_{21} \sim N(\mu_{21}, \sigma_{21}^2)$$

calculate mean and variance

$$\mu_{21} = n(\gamma_1 - \gamma_2 + \rho_J \delta_1 \delta_2 - \delta_2^2) \tag{38}$$

$$\sigma_{21}^2 = (\alpha_1^2 + \alpha_2^2 - 2\rho\alpha_1\alpha_2)T + (\beta_1^2 + \beta_2^2 - 2\rho_H \beta_1 \beta_2) T^{2H} + n(\delta_2^2 + \delta_1^2 - 2\rho_J \delta_1 \delta_2) \tag{39}$$

Let

$$d_{21}=\frac{X_{21}-\mu_{21}}{\sigma_{21}} \tag{40}$$

$$Z_{21}=\frac{Y_{21}-\mu_{21}}{\sigma_{21}}\sim N(0,1) \tag{41}$$

where

$$X_{22}=\ln\frac{S_0^{(2)}}{K}+(r+\frac{1}{2}\alpha_2^2-\lambda\kappa_2)T+\frac{1}{2}\beta_2^2T^{2H} \tag{42}$$

$$Y_{22}=-\alpha_2\tilde{W}_2(T)-\beta_2\tilde{W}_2^H(T)-\sum_{k=1}^{N_T}\tilde{J}_k^{(2)},\ Y_{22}\sim N(\mu_{22},\sigma_{22}^2) \tag{43}$$

calculate mean and variance

$$\mu_{22}=-n(\gamma_2+\delta_2^2),\ \sigma_{11}^2=\alpha_2^2T+\beta_2^2T^{2H}+n\delta_2^2 \tag{44}$$

Let

$$d_{22}=\frac{X_{22}-\mu_{22}}{\sigma_{22}} \tag{45}$$

$$Z_{22}=\frac{Y_{22}-\mu_{22}}{\sigma_{22}}\sim N(0,1) \tag{46}$$

$$\begin{aligned}E_{Q_2}[Y_{21}Y_{22}]&=(\alpha_2^2-\rho\alpha_1\alpha_2)T+(\beta_2^2-\rho_H\beta_1\beta_2)T^{2H}\\&-n^2[\delta_2^2+\rho_J\delta_1\delta_2+(\gamma_2+\delta_2^2)(\gamma_1+\rho_J\delta_1\delta_2)]\end{aligned} \tag{47}$$

$$\tilde{\rho}_2=\frac{E_{Q_2}[Y_{21}Y_{22}]-\mu_{21}\mu_{22}}{\sigma_{21}\sigma_{22}} \tag{48}$$

Finally, calculate I_3 .We have

$$\begin{aligned}&P(\mathcal{A}_3\mid\mathcal{F}_0)\\&=\sum_{n=0}^{\infty}e^{-\lambda_1T}\frac{(\lambda_1T)^n}{n!}P(S_T^{(1)}>K\ \text{ or }\ S_T^{(2)}>K\mid N_T=n,\ \mathcal{F}_0)\\&=\sum_{n=0}^{\infty}e^{-\lambda_1T}\frac{(\lambda_1T)^n}{n!}[1-P(S_T^{(1)}\le K\ \text{ and }\ S_T^{(2)}\le K\mid N_T=n,\ \mathcal{F}_0)\end{aligned} \tag{49}$$

It holds that

$$\begin{aligned}&\{S_T^{(1)}\le K\mid N_T=n,\ \mathcal{F}_0\}\\&\Leftrightarrow\\&\{\ln\frac{K}{S_0^{(1)}}-(r-\frac{1}{2}\alpha_1^2-\lambda\kappa_1)T+\frac{1}{2}\beta_1^2T^{2H}\\&\ge\alpha_1W_1(T)+\beta_1W_1^H(T)+\sum_{k=1}^{N_T}J_k^{(1)}\}\end{aligned}$$

Let $N_T=n$

$$X_{31}=\ln\frac{K}{S_0^{(1)}}-(r-\frac{1}{2}\alpha_1^2-\lambda\kappa_1)T+\frac{1}{2}\beta_1^2T^{2H} \tag{50}$$

$$Y_{31}=\alpha_1W_1(T)+\beta_1W_1^H(T)+\sum_{k=1}^{N_T}J_k^{(1)},\ Y_{31}\sim N(\mu_{31},\sigma_{31}^2) \tag{51}$$

$$\mu_{31}=n\gamma_1,\ \sigma_{31}^2=\alpha_1^2T+\beta_1^2T^{2H}+n\delta_1^2 \tag{52}$$

After that

$$d_{31}=\frac{X_{31}-\mu_{31}}{\sigma_{31}} \tag{53}$$

$$Z_{31}=\frac{Y_{31}-\mu_{31}}{\sigma_{31}}\sim N(0,1) \tag{54}$$

$$\{S_T^{(1)}\le K\mid N_T=n,\ \mathcal{F}_0\}\Leftrightarrow\{X_{31}\ge Y_{31}\}\Leftrightarrow\{d_{31}\ge Z_{31}\}$$

And

$$\begin{aligned}&\{S_T^{(2)}\le K\mid N_T=n,\ \mathcal{F}_0\}\\&\Leftrightarrow\\&\{\ln\frac{K}{S_0^{(2)}}-(r-\frac{1}{2}\alpha_2^2-\lambda\kappa_2)T+\frac{1}{2}\beta_2^2T^{2H}\\&\ge\alpha_2(T)+\beta_2W_2^H(T)+\sum_{k=1}^{N_T}J_k^{(2)}\}\end{aligned}$$

let $N_T=n$

$$X_{32}=\ln\frac{K}{S_0^{(2)}}-(r-\frac{1}{2}\alpha_2^2-\lambda\kappa_2)T+\frac{1}{2}\beta_2^2T^{2H} \tag{55}$$

$$Y_{32}=\alpha_2W_2(T)+\beta_2W_2^H(T)+\sum_{k=1}^{N_T}J_k^{(2)},\ Y_{32}\sim N(\mu_{32},\sigma_{32}^2) \tag{56}$$

calculate mean and variance

$$\mu_{32}=n\gamma_2 \tag{57}$$

$$\sigma_{32}^2=\alpha_2^2T+\beta_2^2T^{2H}+n\delta_2^2 \tag{58}$$

Let

$$d_{32}=\frac{X_{32}-\mu_{32}}{\sigma_{32}} \tag{59}$$

$$Z_{32}=\frac{Y_{32}-\mu_{32}}{\sigma_{32}}\sim N(0,1) \tag{60}$$

$$\{S_T^{(2)}\le K\mid N_T=n,\ \mathcal{F}_0\}\Leftrightarrow\{X_{32}\ge Y_{32}\}\Leftrightarrow\{d_{32}\ge Z_{32}\}$$

$$\tilde{\rho}_3=\frac{E_P[Y_{31}Y_{32}]-\mu_{31}\mu_{32}}{\sigma_{31}\sigma_{32}} \tag{61}$$

With

$$E_P[Y_{31}Y_{32}]=\rho\alpha_1\alpha_2T+\rho_H\beta_1\beta_2T^{2H}+n^2\delta_2^2 \tag{62}$$

$$\sum\nolimits_3=\begin{bmatrix}1&\tilde{\rho}_3\\\tilde{\rho}_3&1\end{bmatrix} \tag{63}$$

Hence

$$
\begin{aligned}
&P(\mathcal{A}_3 \mid \mathcal{F}_0) \\
&= \sum_{n=0}^{\infty} e^{-\lambda_1 T} \frac{(\lambda_1 T)^n}{n!} P(S_T^{(1)} > K \quad \text{or} \quad S_T^{(2)} > K \mid N_T = n,\ \mathcal{F}_0) \\
&= \sum_{n=0}^{\infty} e^{-\lambda_1 T} \frac{(\lambda_1 T)^n}{n!} [1 - P(S_T^{(1)} \le K \quad \text{and} \quad S_T^{(2)} \le K \mid N_T = n,\ \mathcal{F}_0) \\
&= \sum_{n=0}^{\infty} e^{-\lambda_1 T} \frac{(\lambda_1 T)^n}{n!} [1 - \Phi(d_{31}, d_{32}, \textstyle\sum_3)]
\end{aligned}
\tag{64}
$$

Last

$$
I_3 = e^{-rT} K \sum_{n=0}^{\infty} e^{-\lambda_1 T} \frac{(\lambda_1 T)^n}{n!} [1 - \Phi(d_{31}, d_{32}, \textstyle\sum_3)] \tag{65}
$$

Summarizing, the semi-closed analytic formula for the values of European call options on the maximum of two-asset options under the Mixed Fractional Brownian motion model with jumps (JMFBM) are given by:

$$
\begin{aligned}
C_{\max}(S_T^{(1)}, S_T^{(1)}, T; K) = {} & S_0^{(1)} \sum_{n=0}^{\infty} e^{-\lambda_1 T} \frac{(\lambda_1 T)^n}{n!} \Phi(d_{11},\ d_{12}, \textstyle\sum_1) \\
& + S_0^{(2)} \sum_{n=0}^{\infty} e^{-\lambda_2 T} \frac{(\lambda_2 T)^n}{n!} \Phi(d_{21},\ d_{22}, \textstyle\sum_2) \\
& - e^{-rT} K \sum_{n=0}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} [1 - \Phi(d_{31}, d_{32}, \textstyle\sum_3)]
\end{aligned}
\tag{66}
$$

4. Conclusion

In this paper, a semi-closed analytic formula for the pricing of European call options on the maximum of two-asset options under the Mixed Fractional Brownian motion model with jumps (JMFBM) are derived using measurement transformations and Radon–Nikodým derived, which facilitates faster simulation and computation of option pricing.

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