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Valuation of Vulnerable Barrier Options in a Mixed Fractional Brownian Motion Environment

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Abstract: ***The analytical pricing formula of vulnerable barrier option when the underlying asset and counterparty asset follows a geometric mixed fractional Brownian motion, with Hurst parameter $H \in (0,1)$, is considered. The derivation of the close-form pricing formula of the vulnerable barrier option is described in detail.***

Keywords: Vulnerable barrier option, Mix fractional Brownian motion, Measure transformation.

1. Introduction

Barrier options are popular among derivative securities traders, especially in the over-the-counter and the foreign exchange market. This popularity can be attributed to its lower premium and elasticity than vanilla options. These options are activated or extinguished when a barrier variable reaches or breaks a specific level from above or below. Due to the various advantages of barrier option, many scholars at home and abroad have studied its pricing. Kirkby and Aguilar[3] derived the pricing formula for the lookback barrier option by discretizing time.

MFBM is a generalization of FBM, which is a linear combination of Brownian motion and independent fractional Brownian motion. Mounir[4] provided a detailed introduction to the properties and theorems of mixed fractional Brownian motion, as well as related knowledge

The underlying assets of vulnerable options are highly exposed to the risk of becoming worthless or losing their value due to market events or price changes. In the over-the-counter market, options are vulnerable due to the credit risk of the counterparty, which may expose the option holder to the risk of default. After the global financial crisis in 2008,vulnerable option received more and more attention. Cheng and Xu[1][2] derived an analytical pricing formula for vulnerable options under jump diffusion mixed fractional Brownian motion by using actuarial method; Two years later, they used actuarial methods again to derive a vulnerable option pricing formula for a mixed fractional Brownian motion model when the company's liabilities follow a stochastic process. Wang and Zhang[6] derived the pricing formulas for vulnerable barrier options and vulnerable double barrier options based on the reflection principle of Brownian motion. Inspired by it, Zhang and Zhou[7] derived the pricing formula for vulnerable chain options by using the reflection principle and Markov properties of Brownian motion.

The objective of this paper is to derive a closed-form pricing formula for the vulnerable barrier option, in which both the underlying asset and the counterparty asset follow geometric mixed fractional Brownian motion. The arrangement of this article is as follows. Section 2 reviews the properties of the mixed fractional Brownian motion. This is followed by a description of the derivation process of the analytical pricing formula for vulnerable barrier option in Section 3. Section 4 concludes this paper.

2. Preliminaries

In this section, before formally pricing vulnerable barrier options, let's review the definition of mixed fractional Brownian motion and its main properties, which we'll use later in the derivation.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, Q)$ be a complete probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. All the random variables and processes below in this paper are defined on this given probability space.

2.1 Mix Fraction Brownian Motion

Definition 2.1. A mixed fractional Brownian motion $M^H_{\alpha,\beta}(t)$ of parameters α, β and H is a linear combination of Brownian motion $B(t)$ and fractional Brownian motion $B^H(t)$ with Hurst parameter $H \in (0,1)$, defined on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, Q)$ by:

$$M^H_{\alpha,\beta}(t) = \alpha B(t) + \beta B^H(t)$$

where $B(t)$ and $B^H(t)$ are independent, α and β are two real constants such that $(\alpha, \beta) \neq (0,0)$. P is the physical probability measure and the information filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ is generated by (B_τ, B^H_τ) for $\tau \leq t$, which satisfies the usual conditions, such as monotonic increasing and right continuous.

we will list some properties of the MFBM on the following section.

Proposition 2.1. The MFBM $M^H_{\alpha,\beta}(t)$ satisfies the following properties:

(i) $M^H_{\alpha,\beta}(t)$ is a centered Gaussian process with mean zero and the covariance function

$$Cov(M^H(t), M^H(s)) = \alpha^2 (t \wedge s) + \frac{\beta^2}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right),$$

with $s, t > 0$;

(ii) $M^H_{\alpha,\beta}(t)$ is not a Markovian process for $H \in (0,1) \setminus \{1/2\}$;

(iii)The increments of $M^H_{\alpha,\beta}(t)$ are stationary and mixed-self-similar, in the sense that, for any $h > 0$,

$$\left\{M^H_{\alpha,\beta}(ht)\right\} \overset{d}{=} \left\{M^H_{\alpha h^{\frac{1}{2}}, \beta h^H}(t)\right\}; \qquad (0.1)$$

where the notation $\overset{d}{=}$ denotes the random variables on the both sides of the equation (2.1) have the same distributions;

(iv)The increments of the process $M^H_{\alpha,\beta}(t)$ are positively correlated if $H \in (\frac{1}{2}, 1)$; uncorrelated if $H = \frac{1}{2}$ and negatively correlated if $H \in (0, \frac{1}{2})$;

(v)The increments of $M^H_{\alpha,\beta}(t)$ are long-range dependence if and only if $H \in (\frac{1}{2}, 1)$;

(vi)The MFBM $M^H_{\alpha,\beta}(t)$ is equivalent to BM for $H \in (\frac{3}{4}, 1)$.

2.2. Basic Setting of the Pricing Model

We assume S_t and V_t represents the Underlying asset price and the Counterparty asset price, respectively. The stochastic process representation of underlying asset goes down,

$$\frac{dS_t}{S_t} = rdt + \sigma_S dB_S(t) + \sigma_S^H dB_S^H(t) \qquad (0.2)$$

Where S_t is the price of Underlying asset, $B_S(t)$ and $B_S^H(t)$ are Brownian motion and fractional Brownian motion, respectively, H is the Hurst parameter and $H \in (\frac{3}{4}, 1)$; r is the risk-free interest rate. $B_S(t)$ and $B_S^H(t)$ are mutually independent.

The stochastic process representation of Counterparty asset price goes down,

$$\frac{dV_t}{V_t} = rdt + \sigma_V dB_V(t) + \sigma_V^H dB_V^H(t), \qquad (0.3)$$

Where V_t is the price of Counterparty asset, $B_V(t)$ and $B_V^H(t)$ are Brownian motion and fractional Brownian motion, respectively. H is the Hurst parameter and $H \in (\frac{3}{4}, 1)$; r is the risk-free interest rate. Similar to the appeal, $B_V(t)$ and are $B_V^H(t)$ mutually independent. $B_S(t)$ and $B_V(t)$ has correlation coefficient ρ_{SV} , $B_S^H(t)$ and $B_V^H(t)$ has correlation coefficient ρ_{SV}^H .

3. Pricing Vulnerable Barrier Options

As we all known, barrier call option can be divided into four categories, namely up-and-in call option, up-and out call option, down-and-in call option and down-and-out call option. Similarly, there are also four types of barrier put options, which correspond to barrier call options. In the following section, we will take up-and-in vulnerable barrier call option as an example, derive its explicit price formula. We can obtain the explicit price formula of other situations by using the same method.

Theorem 3.1. Assuming that the underlying asset and the counterparty asset meet the mixed fractional Brownian motion model (2.2) and (2.3), the analytical pricing formula of vulnerable barrier option can be expressed as

$$\begin{aligned} C_{ui}(0, S_0, V_0) = \sum_{g=1}^{M} \Big[& S_0 N_{M+2}\left(e_{1,g}^{(1)}, e_{2,g}^{(1)}, \cdots, d_g^{(1)}, e_{g+1,g}^{(1)}, \cdots, e_{M,g}^{(1)},\right. \\ & \left. y_K^{(1)}, y_2^{(1)}; \Sigma_g^{(1)}\right) - Ke^{-rT} N_{M+2}\left(e_{1,g}^{(2)}, e_{2,g}^{(2)}, \cdots, d_g^{(2)},\right. \\ & \left. e_{g+1,g}^{(2)}, \cdots, e_{M,g}^{(2)}, y_K^{(2)}, y_2^{(2)}; \Sigma_g^{(1)}\right) + S_0 V_0 \frac{1-\alpha}{D} e^{rT} \\ & N_{M+2}\left(e_{1,g}^{(3)}, e_{2,g}^{(3)}, \cdots, d_g^{(3)}, e_{g+1,g}^{(3)}, \cdots, e_{M,g}^{(3)}, y_K^{(3)},\right. \\ & \left. y_2^{(3)}; \Sigma_g^{(2)}\right) - KV_0 \frac{1-\alpha}{D} e^{rT} N_{M+2}\left(e_{1,g}^{(4)}, e_{2,g}^{(4)}, \cdots,\right. \\ & \left. d_g^{(4)}, e_{g+1,g}^{(4)}, \cdots, e_{M,g}^{(4)}, y_K^{(4)}, y_2^{(4)}; \Sigma_g^{(2)}\right)\Big] \end{aligned}$$

with $0 = t_0 < t_1 < t_2 < \cdots < t_M = T$ and

$$e_{j,g}^{(p)} = -\frac{\xi_{t_j - t_g}^{(p)}}{\delta_{t_j - t_g}^{(p)}}, \ j = 1, 2, \cdots, g-1, g+1, \cdots, M.$$

$$d_g^{(p)} = \frac{ln\frac{S_0}{B} + \xi_{t_g}^{(p)}}{\delta_{t_g}^{(p)}},$$

$$y_k^{(p)} = \frac{ln\frac{S_0}{K} + \xi_{t_M}^{(p)}}{\delta_{t_M}^{(p)}}, \ p = 1, 2, 3, 4,$$

$$y_2^{(p)} = \frac{ln\frac{V_0}{D^*} + \xi_2^{(p)}}{\delta_2^{(p)}} . p = 1, 2;$$

$$y_2^{(p)} = \frac{ln\frac{D^*}{V_0} - \xi_2^{(p)}}{\delta_2^{(p)}} . p = 3, 4;$$

$$\xi_{t_k}^{(1)} = (r + \frac{1}{2}\sigma_S^2)t_k + \frac{1}{2}(\sigma_S^H)^2 t_k^{2H},$$

$$\xi_2^{(1)} = (r - \frac{1}{2}\sigma_V^2 + \rho_{SV}\sigma_S\sigma_V)T + \left[\rho_{SV}^H \sigma_S^H \sigma_V^H - \frac{1}{2}(\sigma_V^H)^2\right] T^{2H},$$

$$\xi_{t_k}^{(2)} = (r - \frac{1}{2}\sigma_S^2)t_k - \frac{1}{2}(\sigma_S^H)^2 t_k^{2H},$$

$$\xi_2^{(2)} = (r - \frac{1}{2}\sigma_V^2)T - \frac{1}{2}(\sigma_V^H)^2 T^{2H},$$

$$\xi_{t_k}^{(3)} = (r + \frac{1}{2}\sigma_S^2 + \rho_{SV}\sigma_S\sigma_V))t_k + \left[\rho_{SV}^H \sigma_S^H \sigma_V^H + \frac{1}{2}(\sigma_S^H)^2\right] t_k^{2H},$$

$$\xi_2^{(3)} = (r + \frac{1}{2}\sigma_V^2 + \rho_{SV}\sigma_S\sigma_V)T + \left[\rho_{SV}^H \sigma_S^H \sigma_V^H + \frac{1}{2}(\sigma_V^H)^2\right] T^{2H},$$

$$\xi_{t_k}^{(4)} = (r - \frac{1}{2}\sigma_S^2 + \rho_{SV}\sigma_S\sigma_V))t_k + \left[\rho_{SV}^H\sigma_S^H\sigma_V^H - \frac{1}{2}(\sigma_S^H)^2\right]t_k^{2H},$$

$$\xi_2^{(4)} = (r + \frac{1}{2}\sigma_V^2)T + \frac{1}{2}(\sigma_V^H)^2)T^{2H},$$

$$\left(\delta_{t_k}^{(1)}\right)^2 = \left(\delta_{t_k}^{(2)}\right)^2 = \left(\delta_{t_k}^{(3)}\right)^2 = \left(\delta_{t_k}^{(4)}\right)^2 = \sigma_S^2 t_k + \left(\sigma_S^H\right)^2 t_k^{2H},$$

$$\left(\delta_2^{(1)}\right)^2 = \left(\delta_2^{(2)}\right)^2 = \left(\delta_2^{(3)}\right)^2 = \left(\delta_2^{(4)}\right)^2 = \sigma_V^2 T + \left(\sigma_V^H\right)^2 T^{2H},$$

$N_{M+2}(a_1, a_2, \cdots, a_{M+1}; \Sigma_g^{(q)})$ is the standard (M+2) dimensional cumulative normal distribution function with correlation matrix $\Sigma_g^{(q)}$, for $q = 1, 2$.

The correlation matrix can be used to express

$$\Sigma_g^{(p)} = \langle \rho_{u,v}^{pg} \rangle_{(M+1)\times(M+1)} \quad i, j = 1, \cdots, M+1; q = 1, 2. \quad (0.4)$$

Where $\rho_{u,v}^{qg}$ is given by

$$\rho_{u,v}^{2g} = \rho_{v,u}^{2g}$$

$$= \begin{cases} 1, & u = v, \\ \sqrt{\dfrac{\sigma_S^2(t_g - t_v) + (\sigma_S^H)^2(t_g^{2H} - t_v^{2H})}{\sigma_S^2(t_g - t_u) + (\sigma_S^H)^2(t_g^{2H} - t_u^{2H})}}, & 1 \le u \le v \le g-1, \\ \sqrt{\dfrac{\sigma_S^2(t_u - t_g) + (\sigma_S^H)^2(t_u^{2H} - t_g^{2H})}{\sigma_S^2(t_v - t_g) + (\sigma_S^H)^2(t_v^{2H} - t_g^{2H})}}, & g+1 \le u \le v \le M, \\ \sqrt{\dfrac{\sigma_S^2(t_g - t_u) + (\sigma_S^H)^2(t_g^{2H} - t_u^{2H})}{\sigma_S^2 t_g + (\sigma_S^H)^2 t_g^{2H}}}, & 1 \le u \le g-1, v = g, \\ \sqrt{\dfrac{\sigma_S^2(t_g - t_u) + (\sigma_S^H)^2(t_g^{2H} - t_u^{2H})}{\sigma_S^2 T + (\sigma_S^H)^2 T^{2H}}}, & 1 \le u \le g-1, v = M+1, \\ -\sqrt{\dfrac{\sigma_S^2(t_u - t_g) + (\sigma_S^H)^2(t_u^{2H} - t_g^{2H})}{\sigma_S^2 T + (\sigma_S^H)^2 T^{2H}}}, & g+1 \le u \le M, v = M+1, \\ \sqrt{\dfrac{\sigma_S^2 t_g + (\sigma_S^H)^2 t_g^{2H}}{\sigma_S^2 T + (\sigma_S^H)^2 T^{2H}}}, & u = g, v = M+1, \\ \dfrac{\sigma_S\sigma_V\rho_{SV}\sqrt{(t_g - t_u)T} + \sigma_S^H\sigma_V^H\rho_{SV}^H\sqrt{(t_g^{2H} - t_u^{2H})T^{2H}}}{\delta_{(t_g - t_u)}^{(1)}\delta_2^{(1)}}, & 1 \le u \le g-1, v = M+2, \\ -\left(\dfrac{\sigma_S\sigma_V\rho_{SV}\sqrt{(t_u - t_g)T} + \sigma_S^H\sigma_V^H\rho_{SV}^H\sqrt{(t_u^{2H} - t_g^{2H})T^{2H}}}{\delta_{(t_u - t_g)}^{(1)}\delta_2^{(1)}}\right), & g+1 \le u \le M, v = M+2, \\ \dfrac{\sigma_S\sigma_V\rho_{SV}\sqrt{t_g T} + \sigma_S^H\sigma_V^H\rho_{SV}^H\sqrt{t_g^{2H}T^{2H}}}{\delta_{t_g}^{(1)}\delta_2^{(1)}}, & u = g, v = M+2, \\ \dfrac{\sigma_S\sigma_V\rho_{SV}T + \sigma_S^H\sigma_V^H\rho_{SV}^H T^{2H}}{\delta_{t_M}^{(1)}\delta_2^{(1)}}, & u = M+1, v = M+2, \\ 0, & else. \end{cases} \quad (0.5)$$

the $\rho_{u,v}^{2g}$ is given by

$$\rho_{u,v}^{2g} = \rho_{v,u}^{2g} = \begin{cases} -\rho_{u,M+2}^{2g}, & 1 \le u \le g-1, v = M+2, \\ -\rho_{u,M+2}^{2g}, & g+1 \le u \le M, v = M+2, \\ -\rho_{g,M+2}^{2g}, & u = g, v = M+2, \\ -\rho_{M+1,M+2}^{2g}, & u = M+1, v = M+2, \\ \rho_{u,v}^{1g}, & else. \end{cases} \quad (0.6)$$

Proof. The *Itô* formula is used for (2.2) and (2.3), the following equation holds,

$$S_T = S_0 e^{(r - \frac{1}{2}\sigma_S^2)T - \frac{1}{2}(\sigma_S^H)^2 T^{2H} + \sigma_S B_S(T) + \sigma_S^H B_S^H(T)}, \quad (0.7)$$

$$V_T = V_0 e^{(r - \frac{1}{2}\sigma_V^2)T - \frac{1}{2}(\sigma_V^H)^2 T^{2H} + \sigma_V B_V(T) + \sigma_V^H B_V^H(T)}. \quad (0.8)$$

Let $S_t^* = ln\frac{S_t}{S_0}$ and $V_t^* = ln\frac{V_t}{V_0}$, then we have

$$S_t^* = (r - \frac{1}{2}\sigma_S^2)t - \frac{1}{2}(\sigma_S^H)^2 t^{2H} + \sigma_S B_S(t) + \sigma_S^H B_S^H(t) \quad (0.9)$$

$$V_t^* = (r - \frac{1}{2}\sigma_V^2)t - \frac{1}{2}(\sigma_V^H)^2 t^{2H} + \sigma_V B_V(t) + \sigma_V^H B_V^H(t). \quad (0.10)$$

Let's discretize time $[0, T]$ into M segments, we have

$$0 = t_0 \le t_1 \le \cdots \le t_M = T.$$

Under risk-neutral measure Q, the price formula of up-and-in vulnerable barrier call option can be expressed as

$$C_0^{ui}=e^{-rT}E\Big[(S_T-K)^+\Big(I_{(V_T\ge D^*,\max_{0\le t\le T}S_t>B)}+\frac{1-\alpha}{D}V_TI_{(V_T<D^*,\max_{0\ge t\ge T}S_t>B)}\Big)\Big]=e^{-rT}\sum_{g=1}^{M}E\Big[(S_T-K)^+\Big(I_{(V_T\ge D^*)}+\frac{1-\alpha}{D}V_TI_{(V_T<D^*)}\Big)$$

$$\Big(I_{(\max_{m=1,\cdots,M}S(t_m)>B)}\Big)\Big]=e^{-rT}\sum_{g=1}^{M}E\Big[(S_T-K)^+\Big(I_{(V_T\ge D^*)}+\frac{1-\alpha}{D}V_TI_{(V_T<D^*)}\Big)\Big(I_{(S(t_g)>B,S(t_j)\le S(t_g),j\ne g,j=1,2,\cdots,M))}\Big)\Big]=I_1+I_2+I_3+I_4. \quad (0.11)$$

Where

$$I_1=e^{-rT}\sum_{g=1}^{M}E\Big[S_TI_{(V_T\ge D^*,S_T\ge K,S(t_g)>B,S(t_j)\le S(t_g),j\ne g,j=1,2,\cdots,M)}\Big]$$

$$I_2=Ke^{-rT}\sum_{g=1}^{M}E\Big[I_{(V_T\ge D^*,S_T\ge K,S(t_g)>B,S(t_j)\le S(t_g),j\ne g,j=1,2,\cdots,M)}\Big]$$

$$I_3=\frac{1-\alpha}{D}e^{-rT}\sum_{g=1}^{M}E\Big[S_TV_TI_{(V_T<D^*,S_T\ge K,S(t_g)>B,S(t_j)\le S(t_g),j\ne g,j=1,2,\cdots,M)}\Big]$$

$$I_4=\frac{1-\alpha}{D}Ke^{-rT}\sum_{g=1}^{M}E\Big[V_TI_{(V_T<D^*,S_T\ge K,S(t_g)>B,S(t_j)\le S(t_g),j\ne g,j=1,2,\cdots,M)}\Big]$$

We calculate I_1 first. We introduce a new probability measure Q,

$$\frac{dQ}{dQ}=\frac{S_T}{E(S_T)}$$

$$=exp\Big[-\frac{1}{2}\sigma_S^2T-\frac{1}{2}(\sigma_S^H)^2T^{2H}+\sigma_SB_S(T)+\sigma_S^HB_S^H(T)\Big]$$

We use fraction Girsanov theorem[5], we have

$$B_S(t)=B_S(t)-\sigma_St,\qquad B_S^H(t)=B_S^H(t)-\sigma_S^Ht^{2H},$$
$$B_V(t)=B_V(t)-\rho_{SV}\sigma_St,\ B_V^H(t)=B_V^H(t)-\rho_{SV}^H\sigma_S^Ht^{2H},$$

$B_S(t)$ and $B_V(t)$ are standard Brownian motion, $B_S^H(t)$ and $B_V^H(t)$ are fractional Brownian motion under probability measure Q.

So under measure Q, (3.7) and (3.8) transforms into

$$S_t^*=(r+\frac{1}{2}\sigma_S^2)t+\frac{1}{2}(\sigma_S^H)^2t^{2H}+\sigma_SB_S(t)+\sigma_S^HB_S^H(t). \quad (0.12)$$

$$V_t^*=(r-\frac{1}{2}\sigma_V^2+\rho_{SV}\sigma_S\sigma_V)t+\Big(\rho_{SV}^H\sigma_S^H\sigma_V^H-\frac{1}{2}(\sigma_V^H)^2\Big)t^{2H}+\sigma_VB_V(t)+\sigma_V^HB_V^H(t). \quad (0.13)$$

$$I_1=e^{-rT}\sum_{g=1}^{M}E\Big[S_TI_{(V_T\ge D^*,S_T\ge K,S(t_g)>B,S(t_j)\le S(t_g),j\ne g,j=1,2,\cdots,M)}\Big]=S_0\sum_{g=1}^{M}Q(V_T\ge D^*,S_T\ge K,S(t_g)>B,S(t_j)\le S(t_g),j\ne g,j=1,2,\cdots,M)$$

$$=S_0\sum_{g=1}^{M}Q(V_T^*\ge ln\frac{D^*}{V_0},S_T^*\ge ln\frac{K}{S_0},S_{t_g}^*>ln\frac{B}{S_0},\ S_{t_j}^*-S_{t_g}^*\le 0,j\ne g,j=1,2,\cdots,M)$$

$$=S_0\sum_{g=1}^{M}Q\Big[-\frac{Z_2^{(1)}}{\delta_2^{(1)}}\le\frac{\ln\frac{V_0}{D^*}+\xi_2^{(1)}}{\delta_2^{(1)}},-\frac{Z_{t_M}^{(1)}}{\delta_{t_M}^{(1)}}\le\frac{ln\frac{S_0}{K}+\xi_{t_M}^{(1)}}{\delta_{t_M}^{(1)}},\quad -\frac{Z_{t_g}^{(1)}}{\delta_{t_g}^{(1)}}<\frac{ln\frac{S_0}{B}+\xi_{t_g}^{(1)}}{\delta_{t_g}^{(1)}},\frac{Z_{t_j-t_g}^{(1)}}{\delta_{|t_j-t_g|}^{(1)}}\le-\frac{\xi_{t_j-t_g}^{(1)}}{\delta_{|t_j-t_g|}^{(1)}},j\ne g,\ j=1,2,\cdots,M\Big]$$

with

$$Z_{t_k}^{(1)}=\sigma_SB_S(t_k)+\sigma_S^HB_S^H(t_k),$$

$$\xi_{t_k}^{(1)}=(r+\frac{1}{2}\sigma^2)t_k+\frac{1}{2}(\sigma_S^H)^2t_k^{2H},$$

$$\left(\delta_{t_k}^{(1)}\right)^2=\sigma_S^2t_k+\left(\sigma_S^H\right)^2t_k^{2H},$$

$$Z_2^{(1)}=\sigma_VB_V(T)+\sigma_V^HB_V^H(T),$$

$$\xi_2^{(1)}=(r-\frac{1}{2}\sigma_V^2+\rho_{SV}\sigma_S\sigma_V)T+(\rho_{SV}^H\sigma_S^H\sigma_V^H-\frac{1}{2}(\sigma_V^H)^2)T^{2H},$$

$$\left(\delta_2^{(1)}\right)^2=\sigma_V^2T+\left(\sigma_V^H\right)^2T^{2H}.$$

therefore,

$$I_1=S_0\sum_{g=1}^{M}Q(X_1^{(1)}\le e_{1,g}^{(1)},X_2^{(1)}\le e_{2,g}^{(1)},\cdots,X_g^{(1)}\le d_g^{(1)},$$
$$X_{g+1}^{(1)}\le e_{g+1,g}^{(1)},\cdots,X_M^{(1)}\le e_{M,g}^{(1)},X_{M+1}^{(1)}\le y_K^{(1)},X_{M+2}^{(1)}\le y_2^{(1)})$$
$$=S_0\sum_{g=1}^{M}N_{M+2}(e_{1,g}^{(1)},e_{2,g}^{(1)},\cdots,d_g^{(1)},e_{g+1,g}^{(1)},\cdots,e_{M,g}^{(1)},y_K^{(1)},y_2^{(1)};\Sigma_g^{(1)}) \quad (0.14)$$

where,

$$X_1^{(1)}=-\left(\frac{Z_{t_g-t_1}^{(1)}}{\delta_{t_g-t_1}^{(1)}}\right),\ X_2^{(1)}=-\left(\frac{Z_{t_g-t_2}^{(1)}}{\delta_{t_g-t_2}^{(1)}}\right),\cdots,$$

$$X_g^{(1)}=-\left(\frac{Z_{t_g}^{(1)}}{\delta_{t_g}^{(1)}}\right),\quad X_{g+1}^{(1)}=\frac{Z_{t_{g+1}-t_g}^{(1)}}{\delta_{t_{g+1}-t_g}^{(1)}},\cdots,$$

$$X_M^{(1)}=\frac{Z_{t_M-t_g}^{(1)}}{\delta_{t_M-t_g}^{(1)}},\quad X_{M+1}^{(1)}=-\left(\frac{Z_{t_M}^{(1)}}{\delta_{t_M}^{(1)}}\right),$$

$$X_{M+2}^{(1)}=-\left(\frac{Z_2^{(1)}}{\delta_2^{(1)}}\right),\ e_{j,g}^{(1)}=-\frac{\xi_{t_j-t_g}^{(1)}}{\delta_{|t_j-t_g|}^{(1)}},$$

$$d_g^{(1)}=\frac{ln\frac{S_0}{B}+\xi_{t_g}^{(1)}}{\delta_{t_g}^{(1)}},\ y_K^{(1)}=\frac{ln\frac{S_0}{K}+\xi_{t_M}^{(1)}}{\delta_{t_M}^{(1)}},$$

$$y_2^{(1)}=\frac{ln\frac{V_0}{D^*}+\xi_2^{(1)}}{\delta_2^{(1)}},\ j=1,2,\cdots,g-1,g+1,\cdots,M.$$

We taking $1\le u\le v\le g-1$ as an example, then we have

$$\rho_{u,v}^{1g}=E(X_u^{(1)}X_v^{(1)})=\sqrt{\frac{\sigma_S^2(t_g-t_v)+(\sigma_S^H)^2(t_g^{2H}-t_v^{2H})}{\sigma_S^2(t_g-t_u)+(\sigma_S^H)^2(t_g^{2H}-t_u^{2H})}},$$

using the above method repeatedly, we can obtain the correlation matrix $\Sigma_g^{(1)}$.

Next, let's start calculating I_2.

$$I_2 = Ke^{-rT}\sum_{g=1}^{M} E\left[I_{(V_T \geq D^*, S_T \geq K, S(t_g) > B, S(t_j) \leq S(t_g), j\neq g, j=1,2,\cdots,M)} \right]$$

$$= Ke^{-rT}\sum_{g=1}^{M} Q(V_T \geq D^*, (S_T \geq K, S_{t_g} > B, S_{t_j} \leq S_{t_g},$$
$$j \neq g, j = 1,2,\cdots,M)$$
$$= Ke^{-rT}\sum_{g=1}^{M} N_{M+2}(e_{1,g}^{(2)},\cdots,d_g^{(2)},\cdots,e_{M,g}^{(2)},y_K^{(2)},y_2^{(2)};\Sigma_g^{(1)}) \tag{0.15}$$

where

$$e_{j,g}^{(2)} = -\frac{\xi_{t_j-t_g}^{(2)}}{\delta_{t_j-t_g}^{(2)}}, j = 1,2,\cdots,g-1,g+1,\cdots,M.$$

$$d_g^{(2)} = \frac{ln\frac{S_0}{B} + \xi_{t_g}^{(2)}}{\delta_{t_g}^{(2)}}, y_K^{(2)} = \frac{ln\frac{S_0}{K} + \xi_{t_M}^{(2)}}{\delta_{t_M}^{(2)}},$$

$$y_2^{(2)} = \frac{ln\frac{V_0}{D^*} + \xi_2^{(2)}}{\delta_2^{(2)}},$$

$$\xi_{t_k}^{(2)} = (r - \frac{1}{2}\sigma_S^2)t_k - \frac{1}{2}(\sigma_S^H)^2 t_k^{2H},$$

$$\left(\delta_{t_k}^{(2)}\right)^2 = \sigma_S^2 t_k + \left(\sigma_S^H\right)^2 t_k^{2H}, k = 1,2.\cdots,M,$$

$$\xi_2^{(2)} = (r - \frac{1}{2}\sigma_V^2)T - \frac{1}{2}(\sigma_V^H)^2 T^{2H},$$

$$\left(\delta_2^{(2)}\right)^2 = \sigma_V^2 T + \left(\sigma_V^H\right)^2 T^{2H}.$$

In order to calculate I_3, we introduce a new probability measure $\bar{Q}$,

$$\frac{d\bar{Q}}{dQ} = \frac{S_T V_T}{E(S_T V_T)},$$

we use fraction Girsanov theorem, under probability measure $\bar{Q}$, we have

$$\bar{B}_S(t) = B_S(t) - \sigma_S t - \rho_{SV}\sigma_V t,$$
$$\bar{B}_S^H(t) = B_S^H(t) - \sigma_S^H t^{2H} - \rho_{SV}^H \sigma_V^H t^{2H},$$
$$\bar{B}_V(t) = B_V(t) - \sigma_V t - \rho_{SV}\sigma_S t,$$
$$\bar{B}_V^H(t) = B_V^H(t) - \sigma_V^H t^{2H} - \rho_{SV}^H \sigma_S^H t^{2H},$$

$\bar{B}_S(t)$ and $\bar{B}_V(t)$ are standard Brownian motion, $\bar{B}_S^H(t)$ and $\bar{B}_V^H(t)$ are fractional Brownian motion,

So under measure Q, (3.7) and (3.8) transforms into

$$\bar{S}_t^* = (r + \frac{1}{2}\sigma_S^2 + \rho_{SV}\sigma_S\sigma_V)t + \left(\rho_{SV}^H\sigma_S^H\sigma_V^H + \frac{1}{2}(\sigma_S^H)^2\right)t^{2H} + \sigma_S\bar{B}_S(t) + \sigma_S^H\bar{B}_S^H(t) \tag{0.16}$$

$$\bar{V}_t^* = (r + \frac{1}{2}\sigma_V^2 + \rho_{SV}\sigma_S\sigma_V)t + \left(\rho_{SV}^H\sigma_S^H\sigma_V^H + \frac{1}{2}(\sigma_V^H)^2\right)t^{2H} + \sigma_V\bar{B}_V(t) + \sigma_V^H\bar{B}_V^H(t) \tag{0.17}$$

$$I_3 = \frac{1-\alpha}{D}e^{-rT}\sum_{g=1}^{M} E\left[S_T V_T I_{(V_T < D^*, S_T \geq K, S(t_g) > B, S(t_j) \leq S(t_g), j\neq g, j=1,2,\cdots,M)}\right]$$
$$= \frac{1-\alpha}{D}e^{rT}S_0V_0\sum_{g=1}^{M}\bar{Q}(V_T < D^*, S_T \geq K, S_{t_g} > B, S_{t_j} \leq S_{t_g}, j\neq g, j=1,2,\cdots,M)\, j\neq g, j=1,2,\cdots,M) \tag{0.18}$$
$$= \frac{1-\alpha}{D}e^{rT}S_0V_0\sum_{g=1}^{M} N_{M+2}(e_{1,g}^{(3)},e_{2,g}^{(3)},\cdots,d_g^{(13)},e_{g+1,g}^{(3)},\cdots,e_{M,g}^{(3)},y_K^{(1)},y_2^{(1)};\Sigma_g^{(2)}),$$

with

$$e_{j,g}^{(3)} = -\frac{\xi_{t_j-t_g}^{(3)}}{\delta_{t_j-t_g}^{(3)}}, j = 1,2,\cdots,g-1,g+1,\cdots,M.$$

$$d_g^{(3)} = \frac{ln\frac{S_0}{B} + \xi_{t_g}^{(3)}}{\delta_{t_g}^{(3)}}, y_K^{(3)} = \frac{ln\frac{S_0}{K} + \xi_{t_M}^{(3)}}{\delta_{t_M}^{(3)}},$$

$$y_2^{(3)} = \frac{ln\frac{D^*}{V_0} - \xi_2^{(3)}}{\delta_2^{(3)}},$$

$$\xi_{t_k}^{(3)} = (r + \frac{1}{2}\sigma_S^2 + \rho_{SV}\sigma_S\sigma_V)t_k + \left(\rho_{SV}^H\sigma_S^H\sigma_V^H + \frac{1}{2}(\sigma_S^H)^2\right)t_k^{2H},$$

$$\left(\delta_{t_k}^{(3)}\right)^2 = \sigma_S^2 t_k + \left(\sigma_S^H\right)^2 t_k^{2H},$$

$$\xi_2^{(3)} = (r + \frac{1}{2}\sigma_V^2 + \rho_{SV}\sigma_S\sigma_V)T + (\rho_{SV}^H\sigma_S^H\sigma_V^H + \frac{1}{2}(\sigma_V^H)^2)T^{2H},$$

$$\left(\delta_2^{(3)}\right)^2 = \sigma_V^2 T + \left(\sigma_V^H\right)^2 T^{2H}.$$

We can calculate $\Sigma_g^{(2)}$ by using the method that use to calculate $\Sigma_g^{(1)}$.

Finally, we calculate I_4. We introduce a new probability measure $\hat{Q}$,

$$\frac{d\hat{Q}}{dQ} = \frac{V_T}{E(V_T)}$$

we use fraction Girsanov theorem, under probability measure $\hat{Q}$, we have

$$\hat{B}_S(t) = B_S(t) - \rho_{SV}\sigma_V t,\ \hat{B}_S^H(t) = B_S^H(t) - \rho_{SV}^H\sigma_V^H t^{2H},$$
$$\hat{B}_V(t) = B_V(t) - \sigma_V t,\quad \hat{B}_V^H(t) = B_V^H(t) - \sigma_V^H t^{2H},$$

$\hat{B}_S(t)$ and $\hat{B}_V(t)$ are standard Brownian motion, $\hat{B}_S^H(t)$ and $\hat{B}_V^H(t)$ are fractional Brownian motion.

So under measure $\hat{Q}$, (3.7) and (3.8) transforms into

$$\hat{S}_t^* = (r - \frac{1}{2}\sigma_S^2 + \rho_{SV}\sigma_S\sigma_V)t + \left(\rho_{SV}^H\sigma_S^H\sigma_V^H - \frac{1}{2}(\sigma_S^H)^2\right)t^{2H} + \sigma_S\hat{B}_S(t) + \sigma_S^H\hat{B}_S^H(t). \tag{0.19}$$

$$\hat{V}_t^* = (r + \frac{1}{2}\sigma_V^2)t + \frac{1}{2}(\sigma_V^H)^2 t^{2H} + \sigma_V \hat{B}_V(t) + \sigma_V^H \hat{B}_V^H(t). \quad (0.20)$$

$$\begin{aligned} I_4 &= \frac{1-\alpha}{D} K e^{-rT} \sum_{g=1}^{M} E\left[V_T I_{(V_T < D^*, S_T \geq K, S(t_g) > B, S(t_j) \leq S(t_g), j \neq g, j=1,2,\cdots,M)} \right] \\ &= \frac{1-\alpha}{D} K V_0 \sum_{g=1}^{M} \hat{Q}(V_T < D^*, S_T \geq K, S_{t_g} > B, S_{t_j} \leq S_{t_g}, j \neq g, j = 1,2,\cdots,M) \quad (0.21) \\ &= \frac{1-\alpha}{D} K V_0 \sum_{g=1}^{M} N_{M+2}(e_{1,g}^{(4)}, e_{2,g}^{(4)}, \cdots, d_g^{(4)}, e_{g+1,g}^{(4)}, \cdots, e_{M,g}^{(4)}, y_K^{(4)}, y_2^{(4)}; \Sigma_g^{(2)}) \end{aligned}$$

with

$$e_{j,g}^{(4)} = -\frac{\xi_{t_j - t_g}^{(4)}}{\delta_{t_j - t_g}^{(4)}}, j = 1,2,\cdots,g-1,g+1,\cdots,M.$$

$$d_g^{(4)} = \frac{ln\frac{S_0}{B} + \xi_{t_g}^{(4)}}{\delta_{t_g}^{(4)}}, y_K^{(4)} = \frac{ln\frac{S_0}{K} + \xi_{t_M}^{(3)}}{\delta_{t_M}^{(3)}},$$

$$y_2^{(4)} = \frac{ln\frac{D^*}{V_0} - \xi_2^{(4)}}{\delta_2^{(4)}},$$

$$\xi_{t_k}^{(4)} = (r - \frac{1}{2}\sigma_S^2 + \rho_{SV}\sigma_S\sigma_V)t_k + \left(\rho_{SV}^H \sigma_S^H \sigma_V^H - \frac{1}{2}(\sigma_S^H)^2\right) t_k^{2H},$$

$$\left(\delta_{t_k}^{(4)}\right)^2 = \sigma_S^2 t_k + \left(\sigma_S^H\right)^2 t_k^{2H},$$

$$\xi_2^{(4)} = (r + \frac{1}{2}\sigma_V^2)T + \frac{1}{2}(\sigma_V^H)^2 T^{2H},$$

$$\left(\delta_2^{(4)}\right)^2 = \sigma_V^2 T + \left(\sigma_V^H\right)^2 T^{2H}.$$

The proof is over.

4. Conclusion

In this work, we have reviewed the relevant knowledge of mixed fractional Brownian motion. Then we taking up-and-in call option as an example, the Closed-form formulas for the vulnerable barrier option in a mixed fractional Brownian motion environment is provided explicitly.

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