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Valuation of Reset Option with Multiple Reset Features under Mixed Fractional Brownian Motion Model

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Abstract: ***In this paper, we consider the pricing of reset option when the underlying asset follows a mixed fractional Brownian motion and the Hurst parameter H ∈(0,1). Using quasi-martingale method, measure transformation and Girsanov theorem, the analytical expression of reset option pricing under risk neutral measure is obtained. The main contribution of this paper is to provide the closed-form pricing formula of reset option with strike resets and predetermined reset dates.***

Keywords: Reset Option, Mixed Fractional Brownian Motion, Multiple Reset Features**.**

1. Introduction

As financial markets have developed, many emerging securities markets have introduced option products offering more flexible trading methods and prices, termed as new types of options, to cater to the hedging needs of various clients. These options are characterized by the final return of the option depends not only on the price of the underlying asset on the expiration date of the option, but also on the process of the price of the underlying asset during the whole validity period of the option.

Hence, they are commonly referred to as "path-dependent options." Reset option, a kind of path dependent options, whose strike price will be reset to a new strike price only on the predetermined reset dates if the price of the underlying asset is lower than one of the strike resets. The reset function in reset options can protect investors when stock prices fall, its play an important role in portfolio insurance. However, there are not many articles research reset option in the academic literature. In 1997, Gray and Whaley[1] researched the pricing and risk characteristics of cyclical reset put warrants. Subsequently, they offered a closed form solution for reset options with a single reset date (Gray and Whaley, 1999[2]). Under the risk neutral framework, Cheng and Zhang[3] derived the explicit closed form pricing formula of multi variable normal distribution. Liao and Wang0 considered a general case of reset option, assume the S_t denotes the underlying asset at time t, K_0 denotes the initial price of the reset option, the payoff at maturity T of reset call option is usually set as

$$V(T) = max\{S_T - K^*, 0\} = (S_T - K^*)^+ \tag{1}$$

where K^* is defined by

$$K^* = \begin{cases} K_0, if\ min\left[S_{t_1}, S_{t_2}, \dots, S_{t_m}\right] \geq D_1 \\ K_j, if\ D_j > min\left[S_{t_1}, S_{t_2}, \dots, S_{t_m}\right] \geq D_{j+1} \\ K_d, if\ D_d > min\left[S_{t_1}, S_{t_2}, \dots, S_{t_m}\right] \end{cases} \tag{2}$$

and $0 < t_1 < t_2 < \cdots < t_m < T$ are m predetermined reset dates, $K_j, j = 1, 2, \cdots, d$, are the reset strike prices, $D_j, j = 1, 2, \cdots, d$, denotes the strike resets. Liao and Wang[8] derived the exact closed-form formula of the multiple-reset option under the risk-neutral framework.

On the other hand, in the previous researches under the assumption that the risky asset price follows Black-Scholes model (BS). Actually, it is well known that the Geometric Brownian Motion (GBM) assumption of the underlying asset price dynamics in the BS model fails to reflect the true fact that market return exhibits excessive kurtosis, fat tailed distributions and long-term correlation. One way to deal with the problem is to assume that the underlying risky asset price dynamics follow the processes, which driven by the Factional Brownian Motion (FBM). Hu and Oksendal[7] derived the price formula for European options at time $t = 0$. Necula[6] extended the formula for any time $t \in [0, T]$. However, FBM is not a semi martingale, it is difficult to define random integrals. The MFBM is essentially a Gaussian process family consisting of a linear combination of Brownian motion (BM) and Fractional Brownian motion (FBM). Cheridito[4] proved the MFBM is equivalent to the standard BM under the Hurst index condition, so there are no arbitrage opportunities in financial markets. In reality, very little work on the options valuation is considered in the MFBM market. Therefore, in this paper we consider reset option with multiple reset features under the condition that stock prices evolve into FBM models.

The purpose of this paper is to derive the exact closed-form formula for reset option with strike resets and predetermined reset dates under the risk-neutral framework. The summary of this paper is as follows: Section 2 introduces the MFBM model and provide the pricing formula of reset option under the assumption that the underlying asset price process follows the MFBM. The conclusion of this article in Section 3.

2. The Proposed Model and Pricing Reset Option

2.1 Model

Let $(\Omega, \mathcal{F}, Q)$ be a complete probability space with σ - flow. $\mathcal{F}_t$ is the sigma flow generated by B_t^H, and Q represents the risk neutral measure. Now considering that there are two types of assets in the mixed fraction Black Scholes market, let the bond price M_t satisfy the following equation:

$$dM_t = rM_t dt, M_0 = 1 \quad (3)$$

where r is a risk-free interest rate. A stock price satisfies:

$$\frac{dS_t}{S_t} = rdt + \sigma_1 dB_t^H + \sigma_2 dB_t \quad (4)$$

where $\sigma_1, \sigma_2 > 0$, denotes the volatility.

Consider a $\mathcal{F}_T^H$ −measurable European contingent claim ξ which pays $\xi(S_T)$ at the terminal date T. The price $V(t, S)$ of the European contingent claim at any time t is given by

$$V(t, S) = \widetilde{E}_t\left[e^{-r(T-t)}\xi(S_T)\right] \quad (5)$$

where $\tilde{E}$ is a quasi-conditional expectation under measure Q , and $\widetilde{E}_t[\cdot] = \tilde{E}[\cdot\,|\mathcal{F}_t^H]$.

2.2 Pricing Reset Option

Using the Itô formula, for $t < T$,

$$S_T = S_t\, exp\left\{\left(r - \frac{1}{2}\sigma_2^2\right)(T - t) - \frac{1}{2}\sigma_1^2(T^{2H} - t^{2H}) + \sigma_1(B_t^H - B_t^H) + \sigma_2(B_T - B_t)\right\} \quad (6)$$

The payoff at expiry of the reset option with d strike resets and m predecided reset dates can be written as

$$V(T) = (S_T - K_0)^+ I_{A_1} + (S_T - K_1)^+\left(I_{A_2} - I_{A_1}\right) + \cdots + (S_T - K_{d-1})^+\left(I_{A_d} - I_{A_{d-1}}\right) + (S_T - K_d)^+\left(1 - I_{A_d}\right) \quad (7)$$

where $A_j = \left\{min\left[S_{t_1}, S_{t_2}, \cdots, S_{t_m}\right] \geq D_j\right\}, j \in \{1, 2, \ldots, d\}$, and $I(\cdot)$ is an indicator function. Under the risk-neutral probability measure Q, the arbitragefree price of reset option at time t is

$$\begin{aligned} V(t) &= \tilde{E}^Q\left\{e^{-r(T-t)}V(T)|F_t\right\} \\ &= \tilde{E}_t^Q\left\{e^{-r(T-t)}V(T)\right\} \\ &= \textstyle\sum_{l=1}^{d}\tilde{E}_t^Q\left\{e^{-r(T-t)}(S_T - K_{l-1})^+ I_{A_l}\right\} \\ &\quad -\textstyle\sum_{l=1}^{d}\tilde{E}_t^Q\left\{e^{-r(T-t)}(S_T - K_l)^+ I_{A_l}\right\} \\ &\quad +\tilde{E}_t^Q\left\{e^{-r(T-t)}(S_T - K_d)^+\left(1 - I_{A_d}\right)\right\} \end{aligned} \quad (8)$$

where $0 \leq t < t_1 < t_2 < \cdots < t_m < T$.

Therefore, we know that the key to the solution is to compute the following expression:

$$\textstyle\sum_{l=1}^{d}\tilde{E}_t^Q\left\{e^{-r(T-t)}(S_T - K_h)^+ I_{A_l}\right\} \quad (9)$$

where $h = l$ or $l - 1$.

Theorem: The explicit solution to (9) is as follows:

$$\textstyle\sum_{g=1}^{m}\left[S_t N_{m+1}\left(D_g^{l,h}; \Sigma_g\right) - K_h e^{-r(T-t)} N_{m+1}\left(\overline{D}_g^{l,h}; \Sigma_g\right)\right] \quad (10)$$

where $N_{m+1}\left(\cdot; \Sigma_g\right)$ is the $(m + 1) -$ cumulative probability of dimensional multivariate normal distribution with mean vector 0 and covariance matrix Σ , and $D_g^{l,h}$ stands for the gth row of $D^{l,h}$ as follows:

$$D^{l,h} = \begin{pmatrix} a_{l,1} & b_{1,2} & b_{1,3} & & b_{1,m} & d_h \\ b_{2,1} & a_{l,2} & b_{2,3} & \cdots & b_{2,m} & d_h \\ b_{3,1} & b_{3,2} & a_{l,3} & & b_{3,m} & d_h \\ & \vdots & & \ddots & \vdots & \\ b_{m,1} & b_{m,2} & b_{m,3} & \cdots & a_{l,m} & d_h \end{pmatrix} \quad (11)$$

The components of the matrix $D^{l,h}$ are defined as follows:

$$a_{l,g} = \frac{ln\frac{S_t}{D_l} + \left(r + \frac{1}{2}\sigma_2^2\right)(t_g - t) + \frac{1}{2}\sigma_1^2\left(t_g^{2H} - t^{2H}\right)}{\sqrt{\sigma_1^2\left(t_g^{2H} - t^{2H}\right) + \sigma_2^2\left(t_g - t\right)}}$$

$$b_{g,j} = \frac{\left(r + \frac{1}{2}\sigma_2^2\right)(t_j - t_g) + \frac{1}{2}\sigma_1^2\left(t_j^{2H} - t_g^{2H}\right)}{\sqrt{\sigma_1^2\left(t_j^{2H} - t_g^{2H}\right) + \sigma_2^2\left(t_j - t_g\right)}} \quad (12)$$

$$d_h = \frac{ln\frac{S_t}{K_h} + \left(r + \frac{1}{2}\sigma_2^2\right)(T - t) + \frac{1}{2}\sigma_1^2(T^{2H} - t^{2H})}{\sqrt{\sigma_1^2(T^{2H} - t^{2H}) + \sigma_2^2(T - t)}}$$

$\overline{D}_g^{l,h}$ is similarly defined as $D_g^{l,h}$ with the parameters $a_{l,g}$, $b_{g,j}$ and d_h replaced by $\bar{a}_{l,g}$, $\bar{b}_{g,j}$ and $\bar{d}_h$, respectively:

$$\bar{a}_{l,g} = a_{l,g} - \sqrt{\sigma_1^2\left(t_g^{2H} - t^{2H}\right) + \sigma_2^2\left(t_g - t\right)}$$

$$\bar{b}_{g,j} = b_{g,j} - \sqrt{\sigma_1^2\left(t_j^{2H} - t_g^{2H}\right) + \sigma_2^2\left(t_j - t_g\right)} \quad (13)$$

$$\bar{d}_h = d_h - \sqrt{\sigma_1^2(T^{2H} - t^{2H}) + \sigma_2^2(T - t)}$$

and the correlation matrix $\Sigma_g = \left(\rho_{k,j}^g\right)_{(m+1)\times(m+1)}$, $k, j = 1, 2, \cdots, m + 1$, where $\rho_{k,j}^g$ is given by

$$\rho_{k,j}^g = \rho_{j,k}^g = \begin{cases} 1, k = j \\ \sqrt{\frac{\sigma_1^2\left(t_g^{2H} - t_j^{2H}\right) + \sigma_2^2(t_g - t_j)}{\sigma_1^2\left(t_g^{2H} - t_k^{2H}\right) + \sigma_2^2(t_g - t_k)}}, 1 \leq k < j \leq g - 1, \\ \sqrt{\frac{\sigma_1^2\left(t_k^{2H} - t_g^{2H}\right) + \sigma_2^2(t_k - t_g)}{\sigma_1^2\left(t_j^{2H} - t_g^{2H}\right) + \sigma_2^2(t_j - t_g)}}, g + 1 \leq k < j \leq m \\ -\sqrt{\frac{\sigma_1^2\left(t_g^{2H} - t_j^{2H}\right) + \sigma_2^2(t_g - t_j)}{\sigma_1^2\left(t_g^{2H} - t^{2H}\right) + \sigma_2^2(t_g - t)}}, 1 \leq k \leq g - 1, j = g \\ -\sqrt{\frac{\sigma_1^2\left(t_g^{2H} - t_j^{2H}\right) + \sigma_2^2(t_g - t_j)}{\sigma_1^2(T^{2H} - t^{2H}) + \sigma_2^2(T - t)}}, 1 \leq k \leq g - 1, j = m + 1 \\ \sqrt{\frac{\sigma_1^2\left(t_j^{2H} - t_g^{2H}\right) + \sigma_2^2(t_j - t_g)}{\sigma_1^2(T^{2H} - t^{2H}) + \sigma_2^2(T - t)}}, g + 1 \leq k \leq m, j = m + 1 \\ \sqrt{\frac{\sigma_1^2\left(t_g^{2H} - t^{2H}\right) + \sigma_2^2(t_g - t)}{\sigma_1^2(T^{2H} - t^{2H}) + \sigma_2^2(T - t)}}, k = g, j = m + 1 \\ 0, otherwise \end{cases} \quad (14)$$

Proof. Since

$$\begin{aligned} &\tilde{E}_t^Q\left\{e^{-r(T-t)}(S_T - K_h)^+ I_{A_l}\right\} \\ &= \tilde{E}_t^Q\left\{e^{-r(T-t)}S_T I_{(A_l \cap \{S_T \geq K_h\})}\right\} \\ &\quad -K_h\tilde{E}_t^Q\left\{e^{-r(T-t)} I_{(A_l \cap \{S_T \geq K_h\})}\right\} \\ &= \tilde{E}_t^Q\left\{e^{-r(T-t)}S_T I_{(A_l \cap \{S_T \geq K_h\})}\right\} \\ &\quad -K_h e^{-r(T-t)} Q\left(A_l \cap \{S_T \geq K_h\}\right) \end{aligned} \quad (15)$$

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we choose the stock price S as numeraire and switch measure Q to Q_1, the Radon-Nikodym derivatized as follows:

$$\frac{\boldsymbol{d}Q_1}{\boldsymbol{d}Q} = e^{-r(T-t)}\frac{S_T}{S_t}$$
$$= e^{-\frac{1}{2}\sigma_1^2\left(T^{2H}-t^{2H}\right)-\frac{1}{2}\sigma_2^2(T-t)+\sigma_1\left(B_T^H-B_t^H\right)+\sigma_2(B_T-B_t)} \qquad (16)$$

By Girsanov's theorem, defined by

$$\tilde{B}_t^H = \tilde{B}_t^H - \sigma_1 t^{2H}$$
$$\tilde{B}_t = B_t - \sigma_2 t \qquad (17)$$

$\tilde{B}_t^H$ is a standard Fraction Brownian motion under the Q_1, and $\tilde{B}_t$ is also a standard Brownian motion under the Q_1. Then we can rewrite the (15) as follows:

$$\tilde{E}_t^Q\left\{e^{-r(T-t)}(S_T-K_h)^+ I_{A_l}\right\}$$
$$= S_t Q_1(A_l \cap \{S_T \geq K_h\}) \qquad (18)$$
$$-K_h e^{-r(T-t)} Q(A_l \cap \{S_T \geq K_h\})$$

First, we have

$$Q_1(A_l \cap \{S_T \geq K_h\})$$
$$= Q_1\left(\left\{min\left[S_{t_1}, S_{t_2}, \cdots, S_{t_m}\right] \geq D_l\right\} \cap \{S_T \geq K_h\}\right)$$
$$= \sum_{g=1}^m Q_1\left(S_{t_g} \geq D_l, S_{t_j} \geq S_{t_g}, j \neq = 1, 2, \cdots\ m, S_T \geq K_h\right)$$
$$= \sum_{g=1}^m Q_1\left(ln\, S_{t_g} \geq ln\, D_l\,, ln\, S_{t_j} \geq ln\, S_{t_g}\,, j \neq g, j = 1, 2, \cdots, m, S_T \geq K_h\right)$$
$$= \sum_{g=1}^m Q_1\left(-\sigma_1\left(\tilde{B}_{t_g}^{2H} - \tilde{B}_t^{2H}\right) - \sigma_2\left(\tilde{B}_{t_g} - \tilde{B}_t\right) \leq ln\frac{S_t}{D_l} + \left(r + \frac{1}{2}\sigma_2^2\right)\left(t_g - t\right) + \frac{1}{2}\sigma_1^2\left(t_g^{2H} - t^{2H}\right), -\sigma_1\left(\tilde{B}_{t_j}^{2H} - \tilde{B}_{t_g}^{2H}\right) - \sigma_2\left(\tilde{B}_{t_j} - \tilde{B}_{t_g}\right) \leq \left(r + \frac{1}{2}\sigma_2^2\right)\left(t_j - t_g\right) + \frac{1}{2}\sigma_1^2\left(t_j^{2H} - t_g^{2H}\right), j \neq g, j = 1, 2, \cdots, m, -\sigma_1\left(\tilde{B}_T^{2H} - \tilde{B}_t^{2H}\right) - \sigma_2\left(\tilde{B}_T - \tilde{B}_t\right) \leq\ ln\frac{S_t}{K_h} + \left(r + \frac{1}{2}\sigma_2^2\right)(T - t) + \frac{1}{2}\sigma_1^2\left(T^{2H} - t^{2H}\right)\right)$$
$$= \sum_{g=1}^m Q_1\left(Z_1 \leq b_{g,1}, \ldots, Z_{g-1} \leq b_{g,g-1}, Z_g \leq a_{l,g}\,, Z_{g+1} \leq b_{g,g+1}, \ldots, Z_m \leq b_{g,m}, Z_{m+1} \leq d_h\right)$$
$$= \sum_{g=1}^m N_{m+1}\left(D_g^{l,h}; \textstyle\sum_g\right) \qquad (19)$$

where $Z_j, j = 1,2, \ldots, m+1$, are defined by

$$Z_j = \begin{cases} \dfrac{\sigma_1\left(\tilde{B}_{t_g}^{2H} - \tilde{B}_{t_j}^{2H}\right) + \sigma_2\left(\tilde{B}_{t_g} - \tilde{B}_{t_j}\right)}{\sqrt{\sigma_1^2\left(t_g^{2H} - t_j^{2H}\right) + \sigma_2^2\left(t_g - t_j\right)}}, j = 1,2, \ldots, g-1 \\ -\dfrac{\sigma_1\left(\tilde{B}_{t_g}^{2H} - \tilde{B}_t^{2H}\right) + \sigma_2\left(\tilde{B}_{t_g} - \tilde{B}_t\right)}{\sqrt{\sigma_1^2\left(t_g^{2H} - t^{2H}\right) + \sigma_2^2\left(t_g - t\right)}}, j = g \\ -\dfrac{\sigma_1\left(\tilde{B}_{t_j}^{2H} - \tilde{B}_{t_g}^{2H}\right) + \sigma_2\left(\tilde{B}_{t_j} - \tilde{B}_{t_g}\right)}{\sqrt{\sigma_1^2\left(t_j^{2H} - t_g^{2H}\right) + \sigma_2^2\left(t_j - t_g\right)}}, j = g+1, \ldots, m \\ -\dfrac{\sigma_1\left(\tilde{B}_T^{2H} - \tilde{B}_t^{2H}\right) + \sigma_2\left(\tilde{B}_T - \tilde{B}_t\right)}{\sqrt{\sigma_1^2\left(T^{2H} - t^{2H}\right) + \sigma_2^2(T - t)}}, j = m+1 \end{cases} \qquad (20)$$

and $\sum_g = \left(\rho_{k,j}^g\right)_{(m+1)\times(m+1)}$, $\rho_{k,j}^g = E\left[Z_k Z_j\right]$ .Similarly, we can get

$$Q(A_l \cap \{S_T \geq K_h\}) = \sum_{g=1}^m N_{m+1}\left(\bar{D}_g^{l,h}; \textstyle\sum_g\right) \qquad (21)$$

This completes the proof of Theorem.

Accordingly, the closed-form solution for a reset option with d strike resets and m predecided reset dates $V(t)$ is

$$V(t) = S_t\left\{N(d_d) + \sum_{l=1}^g \sum_{g=1}^m \left[N_{m+1}\left(D_g^{l,l-1}; \textstyle\sum_g\right) - N_{m+1}\left(D_g^{l,l}; \textstyle\sum_g\right)\right]\right\}$$
$$-e^{-r(T-t)}\left\{K_d N\left(\bar{d}_d\right) + \sum_{l=1}^g \sum_{g=1}^m \left[N_{m+1}\left(\bar{D}_g^{l,l-1}; \textstyle\sum_g\right) - N_{m+1}\left(\bar{D}_g^{l,l}; \textstyle\sum_g\right)\right]\right\} \qquad (22)$$

where $N(\cdot)$ is the cumulative probability of the standard normal distribution.

3. Conclusion

In this paper, we have provided the closed-form pricing formula for reset option with strike resets and predetermined reset dates under the mixed fraction Brownian motion model. In addition, the model could be extended to assume that the return dynamics of risky assets follow the jump diffusion model, which is an extension that will be carried out in the future.

References

[1] Gray S F, Whaley R E. Valuing S&P 500 bear market warrants with a periodic reset[J]. Journal of Derivatives, 1997, 5(1): 99-106.

[2] Gray S F, Whaley R E. Reset put options: valuation, risk characteristics, and an application[J]. Australian Journal of Management, 1999, 24(1): 1-20.

[3] Cheng W Y, Zhang S. The analytics of reset options[J]. The Journal of Derivatives, 2000, 8(1): 59-71.

[4] Cheridito P. Mixed fractional Brownian motion[J]. Bernoulli, 2001: 913-934.

[5] Liao S L, Wang C W. Pricing arithmetic average reset options with control variates[J]. Journal of Derivatives, 2002, 10(2): 59-74.
[6] Necula C. Option pricing in a fractional Brownian motion environment[J]. Available at SSRN 1286833, 2002.
[7] Hu Y, Øksendal B. Fractional white noise calculus and applications to finance[J]. Infinite dimensional analysis, quantum probability and related topics, 2003, 6(01): 1-32.
[8] Liao S L, Wang C W. The valuation of reset options with multiple strike resets and reset dates[J]. Journal of Futures Markets: Futures, Options, and Other Derivative Products, 2003, 23(1): 87-107.
[9] Jiang L, Yang D S, Zhang S G. On pricing model of the reset option with n predetermined levels[J]. Journal of Systems Science and Complexity, 2004, 17(1): 137-142.
[10] Dai T S, Fang Y Y, Lyuu Y D. Analytics for geometric average trigger reset options[J]. Applied Economics Letters, 2005, 12(13): 835-840.
[11] Li S J, Li S H, Sun C. A generalization of reset options pricing formulae with stochastic interest rates[J]. Research in International Business and Finance, 2007, 21(2): 119-133.
[12] Deng G H, Lin H Y. Pricing American put option in a fractional black-Scholes model via compound option approximation[J]. Advance in Systems Science and Applications, 2008, 8(3): 447-456.
[13] Deng G. Pricing reset options with multiple reset features under stochastic interest rate and jump diffusion model[C]//Proceedings of the 29th Chinese Control Conference. IEEE, 2010: 5559-5563.

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