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Pricing of Outer Performance Option under a Two-Factor Stochastic Volatility Jump-Diffusion Model

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Abstract: ***This study addresses the pricing of outer performance option within the framework of a two-factor stochastic volatility jump-diffusion model. By integrating martingale theory, partial differential equation (PDE) techniques, the Feynman-Kac theorem, and the Fourier inversion transform, we derive a semi-closed-form pricing formula for outer performance option. The methodology rigorously accounts for both stochastic volatility components and jump risk, thus providing a comprehensive solution to the complex valuation problem.***

Keywords: Double Heston Stochastic Volatility Model, Jump-Diffusion Model, Outer Performance Option, Fourier Inversion Transform.

1. Introduction

In the rapidly developing derivatives market, how to price options reasonably is a very important research topic. Among these derivatives, outer performance option has emerged as essential instruments for financial practitioners, offering a means to hedge risk, optimize portfolios, and engage in speculative trading by capitalizing on the price differential between the two underlying assets. However, due to the interaction of various market factors (such as stochastic volatility and abrupt price jumps), the accurate pricing of outer performance option is facing great challenges.

The development of option pricing models has been marked by a series of significant milestones. Black-Scholes model [1] is a revolutionary contribution to financial economics. By assuming that the underlying asset price follows the geometric Brownian motion and the interest rate and volatility remain unchanged, the basic framework of option valuation is established. However, the Black-Scholes model fails to capture several key empirical characteristics of the real financial market, including volatility clustering, the peak and thick tail of yield distribution, and the occurrence of extreme events. To address these limitations, follow-up studies have focused on incorporating more realistic assumptions.

Researchers have explored two primary directions for enhancing model accuracy. On one hand, the introduction of jump risk into asset price models, as proposed by Merton [2] and further developed by Kou [3], has given rise to jump-diffusion models. These models effectively explain the fat-tailed characteristics of financial returns by integrating discrete jump processes with continuous diffusion dynamics. On the other hand, stochastic volatility models, exemplified by the Heston model [4], have been developed to model the time-varying nature of volatility. Bates [5] merged these two lines of research, combining stochastic volatility and jump-diffusion components to provide a more comprehensive description of asset price movements. Scott [6] extended this framework by considering stochastic interest rates alongside stochastic volatility and jumps, while Duffie et al. [7] proposed affine jump-diffusion models that simplify the mathematical treatment of complex market dynamics. Deng [8] investigated option pricing within a two-factor stochastic volatility framework, taking into account jump-diffusion factors. However, the existing models still can not fully reflect the complexity of the financial market. Previous studies have adopted a single factor framework or only considered one type of uncertainty, ignoring both multiple risk sources and the volatility under the joint action of multiple factors. This limitation has prompted people to develop more complex multifactor models.

This article aims to contribute to the literature by developing a new option pricing framework under the two-factor jump-diffusion model. Specifically, we assume that the price of the underlying asset is influenced by two independent random volatility factors that can capture market fluctuations of different frequencies and various types of market information. The addition of jump diffusion process enables the model to consider sudden price changes caused by unexpected events, thereby providing more realistic asset prices. We derived a semi closed form pricing formula for outer performance option using martingale methods, partial differential equations, and Fourier transform.

2. Model Formulation

Assume there is a frictionless and arbitrage-free financial market that allows continuous trading within the trading period $[0,\mathrm{T}]$. The market includes a risk-free bond and two risky assets (the underlying assets). Let $W_t = (W_{1t}, W_{2t}, B_{1t}, B_{2t})$ be a 4-dimensional standard Brownian motion and $N_t = (N_{1t}^s, N_{2t}^s, N_t^c)$ be a 3-dimensional Poisson process with intensity parameters $(\lambda_1, \lambda_2, \lambda_3)$ on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, Q)$.Considering the jumps in the underlying assets in the market, the risk-neutral measure is not unique, hence we select an appropriate equivalent martingale measure Q . Following the idea of Duffie et al. [7], under the measure Q , the logarithms of the prices of the two underlying assets, where$\mathrm{X}_{1\mathrm{t}} = \ln\mathrm{S}_{1\mathrm{t}}$ and $X_{2t} = lnS_{2t}$, and their volatilities $\mathrm{V}_{1\mathrm{t}}$ and $\mathrm{V}_{2\mathrm{t}}$ satisfy the following system of stochastic differential equations:

$$\begin{cases} dX_{1t} = \left(r_t - \lambda\kappa_1 - \frac{1}{2}V_{1t}\right)dt + \sqrt{V_{1t}}dW_{1t} + J_{1t}^s dN_{1t}^s + J_{1t}^c dN_t^c, \\ dX_{2t} = \left(r_t - \lambda\kappa_2 - \frac{1}{2}V_{2t}\right)dt + \sqrt{V_{2t}}dW_{2t} + J_{2t}^s dN_{2t}^s + J_{2t}^c dN_t^c, \\ dV_{1t} = \alpha_1(\theta_1 - V_{1t})dt + \sigma_1\sqrt{V_{1t}}dB_{1t}, \\ dV_{2t} = \alpha_2(\theta_2 - V_{2t})dt + \sigma_2\sqrt{V_{2t}}dB_{2t}. \end{cases} \quad (1)$$

Assume r_t is the short-term instantaneous interest rate and $r_t = V_{1t} + V_{2t}$, where the correlation coefficient $cov(dW_{1t}, dB_{1t}) = \rho_1 dt, cov(dW_{2t}, dB_{2t}) = \rho_2 dt$, and ρ_1, ρ_2 are constants. The standard Brownian motions $W_t = (W_{1t}, W_{2t}, B_{1t}, B_{2t})$ and the Poisson process $N_t = (N_{1t}^s, N_{2t}^s, N_t^c)$ are independent of each other; the non-negative constants $\alpha_j, \theta_j, \sigma_j$ are respectively the mean reversion speed, long-term mean level, and instantaneous volatility of the two volatilities, and satisfy $2\alpha_j\theta_j \geq \sigma_j^2 (j = 1,2)$. Assume the individual relative jump size sequences $e^{J_{1t}^s}$ of S_{1t} are independently and identically distributed and satisfy $J_{1t}^s \sim N(\mu_{1,s}, \sigma_{1,s}^2)$, the individual relative jump size sequences $e^{J_{2t}^s}$ of S_{2t} are independently and identically distributed and satisfy $J_{2t}^s \sim N(\mu_{2,s}, \sigma_{2,s}^2)$, and the common jump relative size sequence $(e^{J_{1t}^c}, e^{J_{2t}^c})$ of S_{1t} and S_{2t} are independently and identically distributed and satisfy $J_t^c = (J_{1t}^c, J_{2t}^c) \sim N(\mu_{1c}, \mu_{2c}, \sigma_{1c}^2, \sigma_{2c}^2, \rho_c)$. Further assume $J_{1t}^s, J_{2t}^s, J_t^c$ are mutually independent, and independent of the Brownian motion W_t, Poisson process N_t. Let σ-algebra $\mathcal{F}_t$ be the reference family jointly generated by W_t and N_t and $J_{1t}^s, J_{2t}^s, J_t^c$. Denote the joint jump size of X_{1t}, X_{2t} as the variable ϑ , assuming it has the following jump transformation:

$$\vartheta(c_1, c_2) = \frac{\lambda_1\vartheta_1(c_1) + \lambda_2\vartheta_2(c_2) + \lambda_3\vartheta_c(c_1, c_2)}{\lambda},$$

$$\lambda = \lambda_1 + \lambda_2 + \lambda_3,\ \kappa_1 = \vartheta(1,0) - 1,\ \kappa_2 = \vartheta(0,1) - 1,$$

$$\vartheta_1(c) = exp\left(\mu_{1s}c + \frac{1}{2}\sigma_{1s}^2c^2\right), \vartheta_2(c) = exp\left(\mu_{2s}c + \frac{1}{2}\sigma_{2s}^2c^2\right),$$

$$\vartheta_c(c_1, c_2) = exp\left(\mu_{1c}c_{11} + \mu_{2c}c_{12} + \frac{1}{2}\sigma_{1c}^2c_1^2 + \frac{1}{2}\sigma_{2c}^2c_2^2 + \rho_c\sigma_{1c}\sigma_{2c}c_1c_2\right).$$

Let

$$X_{1t} = lnS_{1t}, X_{2t} = lnS_{2t}$$

and define $\psi(x_1, x_2, v_1, v_2, \tau; u_1, u_2, u_3, u_4)$ as the joint characteristic function of the two asset prices:

$$\begin{aligned} &\psi(x_1, x_2, v_1, v_2, \tau; u_1, u_2, u_3, u_4) \\ &= \mathbb{E}^{\mathbb{Q}}\left\{e^{-\int_t^T r_s\, ds + iu_1X_{1T} + iu_2X_{2T} + iu_3V_{1T} + iu_4V_{2T}} \mid \mathcal{F}_t\right\} \\ &= \mathbb{E}^{\mathbb{Q}}\left\{e^{-\int_t^T r_s\, ds + iu_1X_{1T} + iu_2X_{2T} + iu_3V_{1T} + iu_4V_{2T}} \mid X_{1t} = x_1, X_{2t} \right. \\ &\left. = x_2, V_{1t} = v_1, V_{2t} = v_2\right\} \\ &= \mathbb{E}_t^{\mathbb{Q}}\left\{e^{-\int_t^T r_s\, ds + iu_1X_{1T} + iu_2X_{2T} + iu_3V_{1T} + iu_4V_{2T}}\right\}, \end{aligned}$$

Where $\tau = T - t, t \in [0, T], i$ is the imaginary unit with $i = \sqrt{-1}, u_1, u_2, u_3, u_4 \in \mathbb{C}$, and $E_t^Q\{\cdot\}$ denotes the conditional expectation under the measure Q based on the reference filtration $\mathcal{F}_t$.

Proposition 1: Assume the asset prices of the two underlying assets satisfy (1). Then the joint characteristic function is:

$$\psi(x_1, x_2, v_1, v_2, \tau; u_1, u_2, u_3, u_4) = e^{iu_1x_1 + iu_2x_2 + A(\tau, u_1, u_2, u_3, u_4)v_1 + B(\tau, u_1, u_2, u_3, u_4)v_2 + C(\tau, u_1, u_2, u_3, u_4)},$$

where

$$A(\tau) = \frac{1}{\sigma_1^2}\left(a_1(u_1) + c_1(u_1, u_2) - \frac{2c_1(u_1, u_2)}{1 - d_1(u_1, u_2, u_3)e^{-c_1(u_1, u_2)\tau}}\right),$$

$$B(\tau) = \frac{1}{\sigma_2^2}\left(a_2(u_2) + c_2(u_1, u_2) - \frac{2c_2(u_1, u_2)}{1 - d_2(u_1, u_2, u_4)e^{-c_2(u_1, u_2)\tau}}\right),$$

$$\begin{aligned} C(\tau) = \ &\alpha_1\theta_1\Big(\frac{1}{\sigma_1^2}\Big((a_1(u_1) + c_1(u_1, u_2))\tau \\ &-2\ ln\frac{1 - d_1(u_1, u_2, u_3)e^{-c_1(u_1, u_2)\tau}}{1 - d_1(u_1, u_2, u_3)}\Big)\Big) \\ &+\alpha_2\theta_2\Big(\frac{1}{\sigma_2^2}\Big((a_2(u_2) + c_2(u_1, u_2))\tau \\ &-2\ ln\frac{1 - d_2(u_1, u_2, u_4)e^{-c_2(u_1, u_2)\tau}}{1 - d_2(u_1, u_2, u_4)}\Big)\Big) \\ &+(-iu_1\lambda\kappa_1 - iu_2\lambda\kappa_2 + \lambda_1\vartheta_1(iu_1) \\ &+\lambda_2\vartheta_2(iu_2) + \lambda_3\vartheta_c(iu_1, iu_2) - \lambda)\tau, \end{aligned}$$

$$a_1(u_1) = \alpha_1 - iu_1\sigma_1\rho_1,$$

$$b_1(u_1, u_2) = \frac{1}{2}iu_1 + iu_2 - \frac{1}{2}u_1^2 - 1,$$

$$c_1(u_1, u_2) = \sqrt{a_1^2(u_1) - 2\sigma_1^2 b_1(u_1, u_2)},$$

$$d_1(u_1, u_2, u_3) = \frac{iu_3\sigma_1^2 - a_1(u_1) + c_1(u_1, u_2)}{iu_3\sigma_1^2 - a_1(u_1) - c_1(u_1, u_2)},$$

$$a_2(u_2) = \alpha_2 - iu_2\sigma_2\rho_2,$$

$$b_2(u_1, u_2) = \frac{1}{2}iu_2 + iu_1 - \frac{1}{2}u_2^2 - 1,$$

$$c_2(u_1, u_2) = \sqrt{a_2^2(u_2) - 2\sigma_2^2 b_2(u_1, u_2)},$$

$$d_2(u_1, u_2, u_4) = \frac{iu_4\sigma_2^2 - a_2(u_2) + c_2(u_1, u_2)}{iu_4\sigma_2^2 - a_2(u_2) - c_2(u_1, u_2)}.$$

The proof of Proposition 1

According to the semimartingale Itô formula and the Feynman-Kac theorem, function

$$\psi = \psi(x_1, x_2, v_1, v_2, \tau) = \psi(x_1, x_2, v_1, v_2, \tau; u_1, u_2, u_3, u_4),$$

satisfes the following partial differential integral equation:

$$\begin{cases} -\frac{\partial\psi}{\partial\tau} + \left(\frac{1}{2}v_1 + v_2 - \lambda\kappa_1\right)\frac{\partial\psi}{\partial x_1} + \left(v_1 + \frac{1}{2}v_2 - \lambda\kappa_2\right)\frac{\partial\psi}{\partial x_2} \\ +\frac{v_1}{2}\frac{\partial^2\psi}{\partial x_1^2} + \frac{v_2}{2}\frac{\partial^2\psi}{\partial x_2^2} + \alpha_1(\theta_1 - v_1)\frac{\partial\psi}{\partial v_1} + \alpha_2(\theta_2 - v_2)\frac{\partial\psi}{\partial v_2} \\ +\frac{1}{2}\sigma_1^2v_1\frac{\partial^2\psi}{\partial v_1^2} + \frac{1}{2}\sigma_2^2v_2\frac{\partial^2\psi}{\partial v_2^2} + \frac{\partial^2\psi}{\partial x_1\partial v_1}v_1\sigma_1\rho_1 \\ +\frac{\partial^2\psi}{\partial x_2\partial v_2}v_2\sigma_2\rho_2 - (v_1 + v_2)\psi \\ +\lambda_1E_t[\psi(x_1 + J_{1t}^s, x_2, v_1, v_2, \tau) - \psi(x_1, x_2, v_1, v_2, \tau)] \\ +\lambda_2E_t[\psi(x_1, x_2 + J_{2t}^s, v_1, v_2, \tau) - \psi(x_1, x_2, v_1, v_2, \tau)] \\ +\lambda_3E_t[\psi(x_1 + J_t^c, x_2 + J_t^c, v_1, v_2, \tau) - \psi(x_1, x_2, v_1, v_2, \tau)] \\ = 0, \\ \psi(x_1, x_2, v_1, v_2, 0) = e^{iu_1X_{1T} + iu_2X_{2T} + iu_3V_{1T} + iu_4V_{2T}}. \end{cases} \quad (2)$$

According to Duffie et al. [7], equation (2) has the following exponential form solution:

$$\psi(x_1, x_2, v_1, v_2, \tau) = e^{iu_1x_1+iu_2x_2+A(\tau,u_1,u_2,u_3,u_4)v_1+B(\tau,u_1,u_2,u_3,u_4)v_2+C(\tau,u_1,u_2,u_3,u_4)}.$$

Substituting the above expressions into the partial differential integral equation (2):

$$\begin{aligned}&-\left(\frac{\partial A}{\partial \tau}v_1 + \frac{\partial B}{\partial \tau}v_2 + \frac{\partial C}{\partial \tau}\right) + \left(\frac{1}{2}v_1 + v_2 - \lambda\kappa_1\right) iu_1\\ &+\left(v_1 + \frac{1}{2}v_2 - \lambda\kappa_2\right) iu_2 + \frac{v_1}{2}(iu_1)^2 + \frac{v_2}{2}(iu_2)^2\\ &+\alpha_1(\theta_1 - v_1)A + \alpha_2(\theta_2 - v_2)B + \frac{1}{2}\sigma_1^2 v_1 A^2\\ &+\frac{1}{2}\sigma_2^2 v_2 B^2 + iu_1\sigma_1\rho_1 v_1 A + iu_2\sigma_2\rho_2 v_2 B - v_1 - v_2\\ &+\lambda_1\vartheta_1(iu_1) + \lambda_2\vartheta_2(iu_2) + \lambda_3\vartheta_c(iu_1, iu_2) - \lambda = 0.\end{aligned} \tag{3}$$

Let $A(\tau) = A(\tau, u_1, u_2, u_3, u_4)$, $B(\tau) = B(\tau, u_1, u_2, u_3, u_4)$, $C(\tau) = C(\tau, u_1, u_2, u_3, u_4)$ to satisfy the following three ordinary differential equations (ODEs):

$$\begin{cases}\frac{\partial A(\tau)}{\partial \tau} = \frac{1}{2}\sigma_1^2 A^2(\tau) + (iu_1\sigma_1\rho_1 - \alpha_1)A(\tau)\\ \qquad + \frac{1}{2}iu_1 + iu_2 - \frac{1}{2}u_1^2 - 1,\\ A(0) = iu_3.\end{cases} \tag{4}$$

$$\begin{cases}\frac{\partial B(\tau)}{\partial \tau} = \frac{1}{2}\sigma_2^2 B^2(\tau) + (iu_2\sigma_2\rho_2 - \alpha_2)B(\tau)\\ \qquad + \frac{1}{2}iu_2 + iu_1 - \frac{1}{2}u_2^2 - 1,\\ B(0) = iu_4.\end{cases} \tag{5}$$

And

$$\begin{cases}\frac{\partial C(\tau)}{\partial \tau} = \alpha_1\theta_1 A(\tau) + \alpha_2\theta_2 B(\tau) - iu_1\lambda\kappa_1 - iu_2\lambda\kappa_2\\ \qquad +\lambda_1\vartheta_1(iu_1) + \lambda_2\vartheta_2(iu_2)\\ \qquad +\lambda_3\vartheta_C(iu_1, iu_2) - \lambda,\\ C(0) = 0.\end{cases} \tag{6}$$

First, solve Equation (4) and rewrite it as:

$$\begin{cases}\frac{\partial A(\tau)}{\partial \tau} = \frac{1}{2}\sigma_1^2 A^2(\tau) - a_1(u_1)A(\tau) + b_1(u_1, u_2),\\ A(0) = iu_3.\end{cases} \tag{7}$$

Integrate both sides of the first equation in (7) over the domain $[0, \tau]$ simultaneously, and we obtain:

$$\int_0^\tau \frac{dA(s)}{\frac{1}{2}\sigma_1^2 A^2(s) - a_1(u_1)A(s) + b_1(u_1, u_2)} = \tau.$$

Then apply the indefinite integral formula:

$$\int \frac{dx}{ax^2 - bx + c} = \frac{1}{-\sqrt{b^2 - 4ac}} ln\left[\frac{2ax - b + \sqrt{b^2 - 4ac}}{2ax - b - \sqrt{b^2 - 4ac}}\right].$$

Combine with the initial condition $A(0) = iu_3$:

$$\frac{\sigma_1^2 A(\tau) - a_1(u_1) + c_1(u_1, u_2)}{\sigma_1^2 A(\tau) - a_1(u_1) - c_1(u_1, u_2)} = \frac{iu_3\sigma_1^2 - a_1(u_1) + c_1(u_1, u_2)}{iu_3\sigma_1^2 - a_1(u_1) - c_1(u_1, u_2)} e^{-c_1(u_1,u_2)\tau}$$

$$= d_1(u_1, u_2, u_3)e^{-c_1(u_1,u_2)\tau}.$$

By rearrangement, we can obtain:

$$A(\tau) = \frac{1}{\sigma_1^2}\left(a_1(u_1) + c_1(u_1, u_2) - \frac{2c_1(u_1,u_2)}{1 - d_1(u_1,u_2,u_3)e^{-c_1(u_1,u_2)\tau}}\right). \tag{8}$$

Next, solve Equation (5). First, rewrite it as

$$\begin{cases}\frac{\partial B(\tau)}{\partial \tau} = \frac{1}{2}\sigma_2^2 B^2(\tau) - a_2(u_2)B(\tau) + b_2(u_1, u_2),\\ B(0) = iu_4.\end{cases} \tag{9}$$

Using the same method as above, we can obtain:

$$B(\tau) = \frac{1}{\sigma_2^2}\left(a_2(u_2) + c_2(u_1, u_2) - \frac{2c_2(u_1,u_2)}{1 - d_2(u_1,u_2,u_4)e^{-c_2(u_1,u_2)\tau}}\right). \tag{10}$$

Finally, solve Equation (6):

$$C(\tau) = \alpha_1\theta_1\int_0^\tau A(s)ds + \alpha_2\theta_2\int_0^\tau B(s)ds + (-iu_1\lambda\kappa_1 - iu_2\lambda\kappa_2 + \lambda_1\vartheta_1(iu_1) + \lambda_2\vartheta_2(iu_2) + \lambda_3\vartheta_c(iu_1, iu_2) - \lambda)\tau. \tag{11}$$

Substitute $A(\tau)$ and $B(\tau)$ into the two definite integrals on the right - hand side of Equation (11). After tedious calculations, we obtain respectively:

$$\int_0^\tau A(s)\,ds = \frac{1}{\sigma_1^2}\left((a_1(u_1) + c_1(u_1, u_2))\tau - 2\,ln\frac{1 - d_1(u_1,u_2,u_3)e^{-c_1(u_1,u_2)\tau}}{1 - d_1(u_1,u_2,u_3)}\right), \tag{12}$$

$$\int_0^\tau B(s)ds = \frac{1}{\sigma_2^2}\left((a_2(u_2) + c_2(u_1, u_2))\tau - 2\,ln\frac{1 - d_2(u_1,u_2,u_4)e^{-c_2(u_1,u_2)\tau}}{1 - d_2(u_1,u_2,u_4)}\right). \tag{13}$$

According to Equations (8) - (13), Proposition 1 is proved.

3. Pricing of Outer Performance Option

Outer performance option is an option whose exercise is determined by the difference in returns between two different underlying assets, and its payoff depends on the price difference between the two assets at maturity. Specifically, the payoff of outer performance option equals the positive part of the difference between the prices of the two assets. That is, the payoff of the option at maturity T is equal to:

$$V(T, S_{1T}, S_{2T}) = max\{S_{1T} - S_{2T}, 0\}. \tag{14}$$

Next, we derive its pricing formula.

Proposition 2: Given a European outer performance option on two underlying assets maturing at time T that satisfies Model (1), the option price at time t is given by:

$$C(t, x_1, x_2, v_1, v_2, T) = S_{1t}\Pi_1(\tau, x_1, x_2, v_1, v_2) - S_{2t}\Pi_2(\tau, x_1, x_2, v_1, v_2).$$

Where

$$\Pi_j(\tau, x_1, x_2, v_1, v_2) = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} R\left[\frac{e^{-iuX_{2T}}\varphi_j(u)}{iu}\right]du,$$

$$\varphi_1(u) = E_t^{Q_1}[e^{iuX_{1T}}] = E_t\left[\frac{e^{-\int_t^T r_s ds + (iu+1)X_{1T}}}{S_{1t}}\right] = \frac{\psi(\tau,x_1,x_2,v_1,v_2;u-i,0,0,0)}{\psi(\tau,x_1,x_2,v_1,v_2;-i,0,0,0)},$$

$$\varphi_2(u) = E_t^{Q_2}[e^{iuX_{1T}}] = E_t\left[\frac{e^{-\int_t^T r_s ds + iuX_{1T} + X_{2T}}}{S_{2t}}\right] = \frac{\psi(\tau,x_1,x_2,v_1,v_2;u,-i,0,0)}{\psi(\tau,x_1,x_2,v_1,v_2;0,-i,0,0)}.$$

The proof of Proposition 2

According to the risk-neutral pricing principle, we know that:

$$C(t,x_1,x_2,v_1,v_2,T)=\mathbb{E}_t\left[e^{-\int_t^T r_s ds}(S_{1T}-S_{2T})^+ \mid \mathcal{F}_t\right]=\mathbb{E}\left[e^{-\int_t^T r_s ds}S_{1T}1_{(X_{1T}\geq X_{2T})} \mid \mathcal{F}_t\right]-\mathbb{E}\left[e^{-\int_t^T r_s ds}S_{2T}1_{(X_{1T}\geq X_{2T})} \mid \mathcal{F}_t\right]$$
$$=S_{1t}Q_1(X_{1T}\geq X_{2T})-S_{2t}Q_2(X_{1T}\geq X_{2T})=S_{1t}\Pi_1(\tau,x_1,x_2,v_1,v_2)-S_{2t}\Pi_2(\tau,x_1,x_2,v_1,v_2).$$

Where $\Pi_j(\tau,x_1,x_2,v_1,v_2)=Q_j(X_{1T}\geq X_{2T}), j=1,2$ and Q_1,Q_2 are equivalent martingale measures of Q. Their Radon-Nikodym derivatives are respectively:

$$\frac{dQ_1}{dQ}=\frac{e^{-\int_t^T r_s ds}S_{1T}}{S_{1t}},$$
$$\frac{dQ_2}{dQ}=\frac{e^{-\int_t^T r_s ds}S_{2T}}{S_{2t}}.$$

Applying the Fourier inversion method, we obtain:

$$\Pi_j(\tau,x_1,x_2,v_1,v_2)=\frac{1}{2}+\frac{1}{\pi}\int_0^{+\infty}R\left[\frac{e^{-iuX_{2T}}\varphi_j(u)}{iu}\right]du,$$

And

$$\varphi_1(u)=E_t^{Q_1}[e^{iuX_{1T}}]=E_t\left[\frac{e^{-\int_t^T r_s ds+(iu+1)X_{1T}}}{S_{1t}}\right]$$
$$=\frac{\psi(\tau,x_1,x_2,v_1,v_2;u-i,0,0,0)}{\psi(\tau,x_1,x_2,v_1,v_2;-i,0,0,0)},$$
$$\varphi_2(u)=E_t^{Q_2}[e^{iuX_{1T}}]=E_t\left[\frac{e^{-\int_t^T r_s ds+iuX_{1T}+X_{2T}}}{S_{2t}}\right]$$
$$=\frac{\psi(\tau,x_1,x_2,v_1,v_2;u,-i,0,0)}{\psi(\tau,x_1,x_2,v_1,v_2;0,-i,0,0)}.$$

Here, $\mathbb{E}_t^{Q_j}[\cdot]$ denotes the conditional expectation under the probability measure Q_j for $j=1,2$.

4. Conclusion

In this paper, under the assumptions of stochastic interest rates, stochastic volatilities, and with the inclusion of jump-diffusion factors in the two underlying assets, a semi-closed-form analytical solution for outer performance option pricing is derived using characteristic function methods and Fourier inversion transforms. Compared with traditional single-factor models, the proposed model exhibits significant advantages in capturing market dynamics.

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