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Pricing of European Vulnerable Options under the Double Exponential Jump-diffusion Model with Stochastic Volatility and Stochastic Interest Rates

Jing Wang

College of Mathematics and Statistics, Guangxi Normal University, Yanshan, Guilin, China

Abstract: ***In this paper, the pricing formula for European vulnerable options is discussed under the framework of a double-exponential jump-diffusion model with stochastic interest rates and stochastic volatility. By utilizing the characteristic function of a multidimensional stochastic vector and the Fourier-Cosine method, among other techniques, we derive the pricing formula for European vulnerable options.***

Keywords: Double-exponential jump-diffusion model, Vulnerable options, Fourier-Cosine method, Stochastic volatility, Stochastic interest rates**.**

1. Introduction

Vulnerable options refer to options that entail credit risk. Since the outbreak of the global financial crisis in 2008, credit risk has garnered increasing attention from investors. In contrast to exchange-traded options, over-the-counter (OTC) options do not involve margin requirements or clearinghouses to compel the option seller to fulfill their obligations at maturity. Consequently, there is a constant risk of counterparty default, which has drawn significant concern from investors. In recent years, the focus of research for many scholars, both domestically and internationally, has shifted towards determining reasonable pricing methodologies for vulnerable options.

Johnson and Stulz [1] were the first to propose incorporating credit risk into option pricing models, introducing the concept of vulnerable options and deriving an explicit solution for European vulnerable options. Klein [2] expanded on Johnson and Stulz's model by assuming a correlation between the value of the underlying asset and the counterparty's asset, thereby a corresponding option pricing formula. Tian et al. [3] further assumed that both the underlying asset and the company's value follow a jump-diffusion process, and provided an expression for the vulnerable option pricing formula. Ma et al. [4] investigated the pricing of vulnerable European options within the framework of the Hawkes jump-diffusion model.

In most of the aforementioned studies on option pricing models, it is commonly assumed that interest rates and volatility are constants, whereas in reality, both interest rates and volatility fluctuate randomly. To align with real-world conditions, many scholars have begun to explore volatility models and interest rate models, with the CIR stochastic interest rate model and the Heston stochastic volatility model being hot topics of research. Numerous scholars have delved into the pricing of vulnerable options under the scenarios of stochastic interest rates and stochastic volatility. Lee et al. [5] investigated the pricing of vulnerable options under the Heston stochastic volatility model. Wang et al. [6] proposed a dual stochastic volatility model that distinguishes between long-term and short-term volatility, assuming the long-term volatility to be constant and the short-term volatility to follow a mean-reverting process, and derived a pricing formula for vulnerable options. Han [7] utilizing the Fourier-Cosine method, derived a numerical solution for European vulnerable put options under a stochastic volatility double-exponential jump model. Ma et al. [8] examined the impact of stochastic interest rates and stochastic volatility on the pricing formula for vulnerable options.

The structure of this paper is as follows. In Section 2, a model incorporating stochastic interest rates, stochastic volatility, and double-exponential jump diffusion is proposed. In Section 3, the joint characteristic function of the log-asset prices of the underlying asset and the counterparty's asset is derived. In Section 4, the pricing formula for European vulnerable options is deduced using the Fourier-Cosine expansion technique. Finally, conclusions are summarized in Section 5.

2. The Model

Assume there exists an arbitrage-free, frictionless financial market where two freely and continuously tradable assets are available. One of these assets is a risk-free bond, and the other is a risky asset. The trading period is $\{\Omega, F_t, Q\}$ be a complete probability space, where Q is the risk-neutral probability measure, and $W_{1t}, W_{2t}, W_t^v, W_t^r$ are standard Brownian motions defined on this space. Here, $cov(dW_{1t}, dW_t^v) = \rho_1$, $cov(dW_{2t}, dW_t^v) = \rho_2$, W_{1t}, W_{2t}, W_t^r are mutually independent. Under the risk-neutral Q probability measure, the price of the underlying asset S_t, the counterparty's asset price Y_t the stochastic volatility v_t, and the stochastic interest rate r_t satisfy the following stochastic differential equations:

$$\begin{aligned} \frac{dS_t}{S_t} &= (r_t - \lambda m_1)dt + \sqrt{v_t}dW_{1t} + (e^{J_1} - 1)dN_t \\ \frac{dY_t}{Y_t} &= (r_t - \lambda m_2)dt + \sqrt{v_t}dW_{2t} + (e^{J_2} - 1)dN_t \\ dv_t &= k_1(\theta_1 - v_t)dt + \sigma_1\sqrt{v_t}dW_t^v \\ dr_t &= k_2(\theta_2 - r_t)dt + \sigma_2\sqrt{r_t}dW_t^r \end{aligned} \tag{1}$$

Where, $m_1 = \frac{p_1\eta_1}{\eta_1-1} + \frac{q_2\eta_2}{\eta_2+1} - 1$, $m_2 = \frac{p_2\xi_1}{\xi_1-1} + \frac{q_2\xi_2}{\xi_2+1} - 1$. The parameters k_1, θ_1, σ_1 represent the mean-reversion speed, long-term volatility, and volatility of the stochastic volatility process v_t respectively. The parameters k_2, θ_2, σ_2 represent the mean-reversion speed, long-term volatility, and volatility of the stochastic interest rate r_t respectively, and satisfy $2k_1\theta_1 \geq \sigma_1^2$, $2k_2\theta_2 \geq \sigma_2^2$. N_t is a Poisson process with constant intensity λ. The jump amplitudes on the asset price St and the counterparty's asset price Y_t are J_1 and J_2 , respectively. J_1 and J_2 follow asymmetric double exponential distributions, with their probability density functions shown as follows:

$$f(J_1) = p_1\eta_1 e^{-\eta_1 J_1} 1_{\{J_1 \geq 0\}} + q_1\eta_2 e^{\eta_2 J_1} 1_{\{J_1 < 0\}}, \eta_1 > 1, \eta_2 > 0$$

$$f(J_2) = p_2\xi_1 e^{-\xi_1 J_2} 1_{\{J_2 \geq 0\}} + q_2\xi_2 e^{\xi_2 J_2} 1_{\{J_2 < 0\}}, \xi_1 > 1, \xi_2 > 0$$

If the risk-free interest rate r_t follows the CIR model presented in Dynamics, the price $P(t,T)$ of a zero-coupon bond with maturity T can be derived as:

$$P(t,T) = E_t[e^{A(t,T) - F(t,T)r}]$$

where,

$$A(t,T) = -k_2\theta_2 \left\{ \frac{4}{(m-k_2)(m+k_2)} ln \frac{2m + (m+k_2)(e^{m(T-t)} - 1)}{2m} + \frac{2(T-t)}{k_1 - m} \right\},$$

$$F(t,T) = \frac{2(e^{m(T-t)} - 1)}{2m + (m+k_2) + (e^{m(T-t)} - 1)}, m = \sqrt{k_2^2 + 2\sigma_2^2}$$

The forward measure Q^T is defined based on the Radon-Nikodym derivative, that is,

$$\frac{dQ^T}{dQ} = \frac{e^{-\int_t^T r_s ds}}{P(t,T)}$$

Under the measure QT the stochastic interest rate rt satisfies the following stochastic differential equation:

$$dr_t = (k_2\theta_2 - (k_2 + F(t,T)\sigma_2^2)r_t)dt + \theta_2\sqrt{r_t}d\overline{W_t^r}$$

Where $d\overline{W_t^r}$ remains a Brownian motion under measure Q, and it satisfie

$$d\overline{W_t^r} = \sigma_2\sqrt{r_t}F(t,T)dt + dW_t^r$$

Assuming that T represents the maturity date of a vulnerable option, the payoff function of a European vulnerable option can be written as:

$$g(S_T, Y_T) = \alpha(S_T - K)^+ [1_{\{Y_T \geq D*\}} + \frac{(1-\omega)Y_T}{D} 1_{\{Y_T < D*\}}]$$

Where, α = 1 represents the payoff of a call option, and α = −1 represents the payoff of a put option. D∗ denotes the constant default boundary, D represents the total debt of the option counterparty, and ω denotes the company's bankruptcy loss ratio. Therefore, under the risk-neutral measure, the price of a European vulnerable option is given by: $P = E[e^{-\int_t^T r_s ds} g(S_T, Y_T)]$

3. Joint Characteristic Function

Lemma 3.1. S_T and Y_T satisfy the market model (1). Let $x = ln\, S_T$, $y = ln\, Y_T$ then the joint characteristic function of x and y is:

$$\Phi(x, y, v, r, \tau; u_1, u_2) = e^{iu_1x + iu_2y + A_1(\tau,u_1,u_2)v + A_2(\tau,u_1,u_2)r + B(\tau,u_1,u_2)}$$

where,

$$A_1(\tau, u_1, u_2) = \frac{1}{\sigma_1^2}\left[\varsigma_1(u_1,u_2) + \gamma_1(u_1,u_2) - \frac{2\gamma_1(u_1,u_2)}{1 - d_1(u_1,u_2)e^{-\gamma_1(u_1,u_2)\tau}}\right]$$

$$A_2(\tau, u_1, u_2) = \frac{1}{\sigma_2^2}\left[\varsigma_2(u_1,u_2) + \gamma_2(u_1,u_2) - \frac{2\gamma_2(u_1,u_2)}{1 - d_2(u_1,u_2)e^{-\gamma_2(u_1,u_2)\tau}}\right]$$

$$\begin{aligned} B(\tau, u_1, u_2) = & -\lambda(im_1u_1 + im_2u_2)\tau \\ & + \frac{k_1\theta_1\tau}{\sigma_1^2}[\varsigma_1(u_1,u_2) - \gamma_1(u_1,u_2)] \\ & - \frac{2k_1\theta_1}{\sigma_1^2} ln[\frac{1 - d_1(u_1,u_2)e^{-\gamma_1(u_1,u_2)\tau}}{1 - d_1(u_1,u_2)}] \\ & + \frac{k_2\theta_2\tau}{\sigma_2^2}[\varsigma_2(u_1,u_2) - \gamma_2(u_1,u_2)] \\ & - \frac{2k_2\theta_2}{\sigma_2^2} ln[\frac{1 - d_2(u_1,u_2)e^{-\gamma_2(u_1,u_2)\tau}}{1 - d_2(u_1,u_2)}] + \lambda\Lambda(u_1,u_2)\tau \end{aligned}$$

and

$$\begin{aligned} \varsigma_1(u_1,u_2) &= (k_1 - \rho_1\sigma_1 iu_1 - \rho_2\sigma_1 iu_2) \\ \varsigma_2(u_1,u_2) &= k_2 + F(t,T)\sigma_2^2 \\ \beta_1(u_1,u_2) &= -\frac{1}{2}(iu_1 + iu_2 + u_1^2 + u_2^2) \\ \beta_2(u_1,u_2) &= iu_1 + iu_2 \\ \gamma_1(u_1,u_2) &= \sqrt{\varsigma_1{}^2(u_1,u_2) - 2\sigma_1^2\beta_1(u_1,u_2)} \\ \gamma_2(u_1,u_2) &= \sqrt{\varsigma_2{}^2(u_1,u_2) - 2\sigma_2^2\beta_2(u_1,u_2)} \\ d_1(u_1,u_2) &= \frac{-\varsigma_1(u_1,u_2) + \gamma_1(u_1,u_2)}{-\varsigma_1(u_1,u_2) - \gamma_1(u_1,u_2)} \\ d_2(u_1,u_2) &= \frac{-\varsigma_2(u_1,u_2) + \gamma_2(u_1,u_2)}{-\varsigma_2(u_1,u_2) - \gamma_2(u_1,u_2)} \end{aligned}$$

$$\Lambda(u_1,u_2) = (\frac{p_1\eta_1}{\eta_1 - iu_1} + \frac{q_1\eta_2}{\eta_2 + iu_1})(\frac{p_2\xi_1}{\xi_1 - iu_2} + \frac{q_2\xi_2}{\xi_2 + iu_2}) - 1$$

4. Pricing of European Vulnerable Options

Let $x_1 = ln\, S_0$, $x_2 = ln\, Y_0$, $y_1 = ln\, S_T$, $y_2 = ln\, Y_T$. Under the risk-neutral measure Q the price of a European vulnerable option with a payoff function $g(y_1, y_2)$ can be expressed as:

$$\begin{aligned} V(t_0; x_1, x_2) &= E^Q\left[e^{-\int_t^T r_s ds} g(y_1, y_2) \middle| \mathrm{F}_{t_0}\right] \\ &= P(t,T) \iint_{\mathbb{R}^2} g(y_1,y_2) f(y_1,y_2|x_1,x_2)\, dy_1 dy_2 \end{aligned}$$

here, $f(y_{1,}y_2|x_1,x_2)$ is the probability density function of the bivariate random variable (y_1, y_2) given the state (x_1, x_2) under the risk-neutral measure Q. According to Ruijter [9] and Fang [10], the specific expression for $f(y_{1,}y_2|x_1,x_2)$ can be obtained as follows:

$$\begin{aligned} & f(y_{1,}y_2|x_1,x_2) \\ & = \frac{2}{b_1 - a_1}\frac{1}{b_2 - a_2}\sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1}[Re(\,(\Phi(\frac{k_1\pi}{b_1 - a_1}, +\frac{k_2\pi}{b_2 - a_2}) \\ & \quad exp(\,ik_1\pi\frac{x_1 - a_1}{b_1 - a_1} + ik_2\pi\frac{x_2 - a_2}{b_2 - a_2})) \\ & + Re(\,(\Phi(\frac{k_1\pi}{b_1 - a_1}, -\frac{k_2\pi}{b_2 - a_2}) \\ & \quad exp(\,ik_1\pi\frac{x_1 - a_1}{b_1 - a_1} - ik_2\pi\frac{x_2 - a_2}{b_2 - a_2}))] \\ & \quad cos(\,k_1\pi\frac{y_1 - a_1}{b_1 - a_1})\,cos(\,k_2\pi\frac{y_2 - a_2}{b_2 - a_2}) \end{aligned}$$

where $\Phi(u_1, u_2)$ is the characteristic function of $f(y_1, y_2|x_1, x_2)$ and Re{·} represents taking the real part of a complex number. After a series of complex calculations:

$$c_k(x_1, x_2) = \frac{1}{1+(\frac{k\pi}{b-a})^2}[cos(k\pi\frac{x_2-a}{b-a})e^{x_2} - cos(k\pi\frac{x_1-a}{b-a})e^{x_1} + \frac{k\pi}{b-a}sin(k\pi\frac{x_2-a}{b-a})e^{x_2} - \frac{k\pi}{b-a}sin(k\pi\frac{x_1-a}{b-a})e^{x_1}]$$

$$h_k(x_1, x_2) = \begin{cases} \frac{b-a}{k\pi}[sin(k\pi\frac{x_2-a}{b-a}) - sin(k\pi\frac{x_1-a}{b-a})], k \neq 0 \\ (x_2 - x_1), k = 0 \end{cases}$$

When $k_1 = 0$:

$$G^{call}(a_1, b_1) = \frac{1}{2}\{(e^{b_1} - Kb_1) - (K - K\,ln\,K)\}$$

$$G^{put}(a_1, b_1) = \frac{1}{2}\{(K\,ln\,K - K) - (Ka_1 - e^{a_1})\}$$

When $k_1 = 1, \ldots, N-1$

$$G^{call}(a_1, b_1) = c_k(ln\,K, b_1) - Kh_k(ln\,K, b_1)$$

$$G^{put}(a_1, b_1) = Kh_k(a_1, ln\,K) - c_k(a_1, ln\,K)$$

When $k_2 = 0$:

$$G(a_2, b_2) = \frac{1}{2}(b_2 - ln\,D^*) + \frac{(1-\omega)}{D}(D^* - e^{a_2})$$

When $k_2 = 1, \ldots, N-1$:

$$G(a_2, b_2) = h_k(ln\,D^*, b_2) + \frac{(1-\omega)}{D}c_k(a_2, ln\,D^*)$$

The pricing formula for European vulnerable options can be obtained:

$$V^{call}(k_1, k_2) = \frac{2}{b_1-a_1}\frac{2}{b_2-a_2}G^{call}(a_1, b_1)G(a_1, b_1)$$

$$V^{put}(k_1, k_2) = \frac{2}{b_1-a_1}\frac{2}{b_2-a_2}G^{put}(a_1, b_1)G(a_1, b_1).$$

5. Conclusion

This paper investigates the pricing of European vulnerable options under double-exponential jumps with stochastic interest rates and stochastic volatility. Compared with previously studied vulnerable option pricing models, this paper's model primarily employs the Fourier-Cosine method to study vulnerable options within the double-exponential jump-diffusion model, providing a pricing formula for European vulnerable options.

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