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Digital Power Exchange Option Pricing under Non-Affine Stochastic Volatility Models

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Abstract: ***This paper investigates the pricing of digital power exchange options under a non-affine stochastic volatility model. We first derive an approximate characteristic function for the logarithmic price distribution of underlying assets through perturbation analysis of partial differential equations. Subsequently, the analytical expression for digital power exchange options is obtained by employing Fourier transform and its inverse transformation.***

Keywords: Option pricing, Non-affine stochastic volatility, Fast Fourier Transform.

1. Introduction

In 1973, Black and Scholes [1] proposed the Black-Scholes model (hereafter the BS model). However, extensive empirical studies have revealed inconsistencies between the BS model's assumptions — geometric Brownian motion for stock prices, normally distributed returns, and constant volatility — and observed market phenomena such as leptokurtic and heavy-tailed distributions and volatility smiles. To better characterize these stylized features of financial asset returns and volatility dynamics, scholars have extended the BS model by developing influential alternative frameworks. Examples include the constant elasticity of variance model by Cox and Ross [2], the jump-diffusion model by Merton [3], and stochastic volatility models by Hull and White [4], Stein and Stein [5], and Heston [6]. Among these, the Heston model stands as a canonical affine stochastic volatility model, providing closed-form pricing formulas for plain European options and effectively explaining the volatility smile phenomenon. Nevertheless, recent research indicates that the square-root volatility process specification in affine models inadequately captures nonlinear characteristics of financial time series. This limitation has spurred significant academic interest in non-affine stochastic volatility models. Empirical results demonstrate the superior performance of non-affine specifications: comparative analyses reveal that non-affine stochastic volatility models reduce root mean square pricing errors by 25–27% compared to their affine counterparts.

In 1978, Margrabe pioneered the use of partial differential equation (PDE) methods to derive a closed-form pricing formula for exchange options under the Black-Scholes (B-S) framework [7]. Subsequent research on exchange option pricing, coupled with the diversification of financial markets, has led to the development of numerous innovative variants. These include power exchange options, Asian exchange options, and barrier exchange options, among others. These advancements have not only enriched the theoretical landscape of exchange options but also provided robust methodological foundations for their pricing and further innovation in financial engineering.

In non-affine stochastic volatility models, the partial differential equation (PDE) governing the characteristic function of the log-price distribution of the underlying asset is inherently nonlinear, which precludes the derivation of an exact analytical expression for the characteristic function and typically obstructs closed-form solutions for European option prices. To better capture the dynamic behavior of asset prices and develop a computationally efficient numerical method for pricing options under such models, this study proposes an innovative approach. First, we perform a linearization of the nonlinear PDE associated with the characteristic function in non-affine stochastic volatility frameworks. Subsequently, by leveraging Fourier transform techniques and their inverse, we derive an approximate analytical representation of the characteristic function.

2. Market Models and Preliminaries

This methodology not only enhances computational tractability but also bridges the gap between theoretical modeling and practical pricing applications in complex volatility environments. Let $W_t = (W_{1t}^{s_1}, W_{2t}^{s_1}, W_{1t}^{s_2}, W_{2t}^{s_2}, W_{1t}^{v}, W_{2t}^{v})$ denote a 6-dimensional standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$ Due to the presence of jumps in the underlying asset prices, the equivalent martingale measure is not unique. Following the approach of [8] we select an appropriate equivalent martingale measure Q under which the discounted price processes of risky assets become martingales. Consider two risky assets S_{1t} and S_{2t} in the market, with their logarithmic prices denoted by $X_{2t} = ln\, S_{2t}$, respectively. Let V_{1t} and V_{2t} represent the long-term and short-term volatility rates. The measure Q is characterized by the following system of stochastic differential equations:

$$\begin{cases} dX_{1t} = R_t dt + \sigma_{11}\sqrt{V_{1t}}dW_{1t}^{s_1} + \sigma_{12}\sqrt{V_{2t}}dW_{2t}^{s_1}, \\ dX_{2t} = R_t dt + \sigma_{21}\sqrt{V_{1t}}dW_{1t}^{s_2} + \sigma_{22}\sqrt{V_{2t}}dW_{2t}^{s_2}, \\ \quad dV_{1t} = \beta_1(\theta_1 - V_{1t})dt + \sigma_1 V_{1t}^{\gamma/2}dW_{1t}^{v}, \\ \quad dV_{2t} = \beta_2(\theta_2 - V_{2t})dt + \sigma_2 V_{2t}^{\gamma/2}dW_{2t}^{v}. \end{cases} \tag{1}$$

where the correlations between Brownian motions are given by:

$$\mathrm{corr}\langle dW_{1t}^{s_1}, dW_{1t}^{s_2}\rangle = \rho_1, \quad \mathrm{corr}\langle dW_{2t}^{s_1}, dW_{2t}^{s_2}\rangle = \rho_2,$$
$$\mathrm{corr}\langle dW_{1t}^{s_1}, dW_{1t}^{v}\rangle = \rho_{11}, \quad \mathrm{corr}\langle dW_{2t}^{s_1}, dW_{2t}^{v}\rangle = \rho_{12},$$
$$\mathrm{corr}\langle dW_{1t}^{s_2}, dW_{1t}^{v}\rangle = \rho_{21}, \quad \mathrm{corr}\langle dW_{2t}^{s_2}, dW_{2t}^{v}\rangle = \rho_{22}.$$

Here, ρ_{ij} (i,j=1,2) are constants, and all other Brownian motions are mutually independent. The coefficients σ_{i1}, σ_{i2} represent the long-term and short-term volatility coefficients for assets S_{1t} and S_{2t} respectively, and are non-negative. The non-negative constants β_i, θ_i and σ_i represent the mean reversion speed, long-term mean level, and volatility of volatility for the two variance processes, respectively, and satisfy the Feller condition $2\beta_i\theta_i > \sigma_i^2$. Following [9], the risk-free rate R_t is modeled as $R_t = \vartheta_0 + \vartheta_1 V_{1t} + \vartheta_2 V_{2t}$, where ϑ_0, ϑ_1 and ϑ_2 are real constants.

3. Characteristic Function

Under the measure Q the joint discounted conditional characteristic function of the random variables X_{it}, V_{it} i=(1,2), is defined as:

$$\begin{aligned}&\Psi(t, x_{1t}, x_{2t}, v_{1t}, v_{2t}; u_1, u_2, u_3, u_4, T)\\ &= E^Q\left[e^{-r(T-t)} + iu_1 X_{1T} + iu_2 X_{2T} + iu_3 v_{1T} + iu_4 v_{2T} \middle| \mathscr{F}_t\right]\\ &= E^Q\left[e^{-r(T-t)} + iu_1 X_{1T} + iu_2 X_{2T} + iu_3 v_{1T} + iu_4 v_{2T} \middle| x_{1t}, x_{2t}, v_{1t}, v_{2t}\right].\end{aligned}$$

where i is the imaginary unit, $t \in [0, t]$, $u_1, u_2. u_3 \in \mathcal{C}$, and $E^Q[\cdot\, |\mathscr{F}_t]$ denotes the conditional expectation under the probability measure Q.

Theorem 1: Assuming the underlying assets follow the market model (1), the joint characteristic function of the two underlying assets has the following expression:

$$\begin{aligned}&\Psi(t, x_1, x_2, v_1, v_2; u_1, u_2, u_3, u_4, T)\\ &= exp(\, iu_1 x_1 + iu_2 x_2 + A(\tau, \mathbf{u})\\ &\quad + B(\tau, \mathbf{u}) v_1 + C(\tau, \mathbf{u}) v_2)\end{aligned} \tag{2}$$

Where $\tau = T - t$ and $\boldsymbol{u} = (u_1, u_2, u_3, u_4)$. The coefficient functions are defined as:

$$\begin{aligned}A(\tau, \mathbf{u}) = \frac{d_3}{4d_1^2}&[(f_1 + h_1)^2\tau - 4h_1\tau(f_1 + h_1)\\ &-4(f_1 + h_1)\, ln\frac{1-q_1 e^{-h_1\tau}}{1-q_1} + 4h_1^2\tau\\ &+4h_1\, ln\frac{1-q_1 e^{-h_1\tau}}{1-q_1} + \frac{4h_1}{1-q_1} - \frac{4h_1}{1-q_1 e^{-h_1\tau}}]\\ &+\frac{f_3\tau}{2d_1}(f_1 - h_1) - \frac{f_3}{d_1} ln\frac{1-q_1 e^{-h_1\tau}}{1-q_1}\\ &+\frac{d_4}{4d_2^2}[(f_2 + h_2)^2\tau - 4h_2\tau(f_2 + h_2)\\ &-4(f_2 + h_2)\, ln\frac{1-q_2 e^{-h_2\tau}}{1-q_2} + 4h_2^2\tau\\ &+4h_2\, ln\frac{1-q_2 e^{-h_2\tau}}{1-q_2} + \frac{4h_2}{1-q_2} - \frac{4h_2}{1-q_2 e^{-h_2\tau}}]\\ &+\frac{f_4\tau}{2d_2}(f_2 - h_2) - \frac{f_4}{d_2} ln\frac{1-q_2 e^{-h_2\tau}}{1-q_2} + g_3\tau\end{aligned}$$

$$\begin{aligned}B(\tau, \mathbf{u}) &= \frac{1}{2d_1}\left[f_1 + h_1 - \frac{2h_1}{1-q_1 e^{-\tau h_1}}\right],\\ C(\tau, \mathbf{u}) &= \frac{1}{2d_2}\left[f_2 + h_2 - \frac{2h_2}{1-q_2 e^{-\tau h_2}}\right].\end{aligned}$$

with parameters:

$$\begin{aligned}d_1 &= \tfrac{1}{2}\sigma_1^2\gamma\theta_1^{\gamma-1}, d_2 = \tfrac{1}{2}\sigma_2^2\gamma\theta_2^{\gamma-1},\\ d_3 &= \tfrac{1-\gamma}{2}\sigma_1^2\theta_1^{\gamma}, d_4 = \tfrac{1-\gamma}{2}\sigma_2^2\theta_2^{\gamma},\end{aligned}$$

$$f_1 = -\frac{1+\gamma}{2}\theta_1^{\frac{\gamma-1}{2}}\rho_{11}\sigma_{11}\sigma_1 iu_1 - \frac{1+\gamma}{2}\theta_1^{\frac{\gamma-1}{2}}\rho_{21}\sigma_{21}\sigma_1 iu_2 + \beta_1,$$

$$f_2 = -\frac{1+\gamma}{2}\theta_2^{\frac{\gamma-1}{2}}\rho_{12}\sigma_{12}\sigma_2 iu_1 - \frac{1+\gamma}{2}\theta_2^{\frac{\gamma-1}{2}}\rho_{22}\sigma_{22}\sigma_2 iu_2 + \beta_2,$$

$$f_3 = \frac{1-\gamma}{2}\theta_1^{\frac{1+\gamma}{2}}\rho_{11}\sigma_{11}\sigma_1 iu_1 + \frac{1-\gamma}{2}\theta_1^{\frac{1+\gamma}{2}}\rho_{21}\sigma_{21}\sigma_1 iu_2 + \beta_1\theta_1,$$

$$f_4 = \frac{1-\gamma}{2}\theta_2^{\frac{1+\gamma}{2}}\rho_{12}\sigma_{12}\sigma_2 iu_1 + \frac{1-\gamma}{2}\theta_1^{\frac{1+\gamma}{2}}\rho_{22}\sigma_{22}\sigma_2 iu_2 + \beta_2\theta_2,$$

$$g_1 = (iu_1 + iu_2)\vartheta_1 - \tfrac{1}{2}\sigma_{11}^2 u_1^2 - \tfrac{1}{2}\sigma_{21}^2 u_2^2 - \sigma_{11}\sigma_{21}\rho_1 u_1 u_2,$$

$$g_2 = (iu_1 + iu_2)\vartheta_2 - \tfrac{1}{2}\sigma_{12}^2 u_1^2 - \tfrac{1}{2}\sigma_{22}^2 u_2^2 - \sigma_{12}\sigma_{22}\rho_2 u_1 u_2,$$

$$g_3 = \vartheta_0(iu_1 + iu_2),$$

$$h_1 = h_1(u_1, u_2, 0{,}0) = \sqrt{f_1^2(u_1, u_2, 0{,}0) - 4d_1 g_1(u_1, u_2, 0{,}0)},$$

$$h_2 = h_2(u_1, u_2, 0{,}0) = \sqrt{f_2^2(u_1, u_2, 0{,}0) - 4d_2 g_2(u_1, u_2, 0{,}0)},$$

$$q_1 = q_1(u_1, u_2, u_3, 0) = \frac{2d_1 iu_3 - f_1(u_1,u_2,0{,}0) + h_1(u_1,u_2,0{,}0)}{2d_1 iu_3 - f_1(u_1,u_2,0{,}0) - h_1(u_1,u_2,0{,}0)},$$

$$q_2 = q_2(u_1, u_2 \mid 0, u_4) = \frac{2d_2 iu_4 - f_2(u_1,u_2,0{,}0) + h_2(u_1,u_2,0{,}0)}{2d_2 iu_4 - f_2(u_1,u_2,0{,}0) - h_2(u_1,u_2,0{,}0)}.$$

Proof: By applying the semi-martingale Itô formula and Feynman-Kac theorem, the characteristic function Ψsatisfies the following parabolic partial differential equation (PDE) with boundary conditions:

$$\begin{aligned}&\frac{\partial\psi}{\partial t} + \sum_{p=1}^{2}\left(R_t - \tfrac{1}{2}\sigma_{p1}^2 v_1 - \tfrac{1}{2}\sigma_{p2}^2 v_2 - \lambda\kappa_p\right)\frac{\partial\psi}{\partial x_p}\\ &+\sum_{p=1}^{2}\beta_p\left(\theta_p - v_p\right)\frac{\partial\psi}{\partial v_p} + \sum_{p=1}^{2}\tfrac{1}{2}\left(\sigma_{p1}^2 v_1 + \sigma_{p2}^2 v_2\right)\frac{\partial^2\psi}{\partial x_p^2}\\ &+\sum_{p=1}^{2}\tfrac{1}{2}\sigma_p^2 v_p^{\gamma}\frac{\partial^2\psi}{\partial v_p^2} + \sum_{p=1}^{2}\sigma_{1p}\,\sigma_{2p} v_p\rho_p\frac{\partial^2\psi}{\partial x_1\,\partial x_2}\\ &+\sum_{j=1}^{2}\sum_{p=1}^{2}\rho_{pj}\,\sigma_{pj}\sigma_p v_j^{\frac{\gamma+1}{2}}\frac{\partial^2\psi}{\partial x_p\,\partial v_j} = 0.\end{aligned} \tag{3}$$

Since the PDE (3) is nonlinear in v1 and v2 we linearize the terms $v_i^{\frac{\gamma+1}{2}}$ and v_i^{γ} (i=1,2) via first-order Taylor expansions around $v_i = \theta_i$:

$$v_i^{\gamma} \approx (1-\gamma)\theta_i^{\gamma} + \gamma\theta_i^{\gamma-1} v_i \tag{4}$$

$$v_i^{\gamma} \approx (1-\gamma)\theta_i^{\gamma} + \gamma\theta_i^{\gamma-1} v_i \tag{5}$$

Substituting (4) and (5) into (3) yields the linearized PDE:

$$\begin{aligned}&\frac{\partial\psi}{\partial t} + \sum_{p=1}^{2}(\vartheta_0 + \vartheta_1 v_1 + \vartheta_2 v_2 - \tfrac{1}{2}\sigma_{p1}^2 v_1 - \tfrac{1}{2}\sigma_{p2}^2 v_2\\ &-\lambda\kappa_p)\frac{\partial\psi}{\partial x_p} + \sum_{p=1}^{2}\beta_p\left(\theta_p - v_p\right)\frac{\partial\psi}{\partial v_p}\\ &+\sum_{p=1}^{2}\tfrac{1}{2}\left(\sigma_{p1}^2 v_1 + \sigma_{p2}^2 v_2\right)\frac{\partial^2\psi}{\partial x_p^2} + \sum_{p=1}^{2}\tfrac{1}{2}\sigma_p^2[(1-\gamma)\theta_p^{\gamma}\\ &+\gamma\theta_p^{\gamma-1} v_p]\frac{\partial^2\psi}{\partial v_p^2} + \sum_{p=1}^{2}\sigma_{1p}\,\sigma_{2p} v_p\rho_p\frac{\partial^2\psi}{\partial x_1\,\partial x_2}\\ &+\sum_{j=1}^{2}\sum_{p=1}^{2}\rho_{pj}\,\sigma_{pj}\sigma_p\left(\tfrac{1-\gamma}{2}\theta_j^{\frac{\gamma+1}{2}} + \tfrac{1}{2}\theta_j^{\frac{\gamma-1}{2}} v_j\right)\frac{\partial^2\psi}{\partial x_p\,\partial v_j} = 0.\end{aligned} \tag{6}$$

According to [8] the partial differential equation (6) admits an exponential-form solution:

$$\begin{aligned}&\Psi(t,x_1,x_2,v_1,v_2;u_1,u_2,u_3,u_4,T)\\&=exp[iu_1x_1+iu_2x_2+A(\tau,\mathbf{u})+B(\tau,\mathbf{u})v_1+C(\tau,\mathbf{u})v_2]\end{aligned} \tag{7}$$

with boundary conditions:

$$\begin{cases}A(0,\mathbf{u})=0,\\B(0,\mathbf{u})=iu_3,\\C(0,\mathbf{u})=iu_4.\end{cases} \tag{8}$$

Substituting (7) into (8) yields the following system of ordinary differential equations:

$$\begin{aligned}&\frac{\partial B}{\partial t}+\frac{1}{2}\sigma_1^2\gamma\theta_1^{\gamma-1}B^2-(-\frac{1+\gamma}{2}\theta_1^{\frac{\gamma-1}{2}}\rho_{11}\sigma_{11}\sigma_1 iu_1\\&-\frac{1+\gamma}{2}\theta_1^{\frac{\gamma-1}{2}}\rho_{21}\sigma_{21}\sigma_1 iu_2+\beta_1)B+(\vartheta_1-\frac{1}{2}\sigma_{11}^2)iu_1\\&+(\vartheta_2-\frac{1}{2}\sigma_{21}^2)iu_2+\frac{1}{2}\sigma_{11}^2(iu_1)^2+\frac{1}{2}\sigma_{21}^2(iu_2)^2\\&-\sigma_{11}\sigma_{21}\rho_1u_1u_2=0\end{aligned} \tag{9}$$

$$\begin{aligned}&\frac{\partial C}{\partial t}+\frac{1}{2}\sigma_2^2\gamma\theta_2^{\gamma-1}C^2-(-\frac{1+\gamma}{2}\theta_2^{\frac{\gamma-1}{2}}\rho_{12}\sigma_{12}\sigma_2 iu_1\\&-\frac{1+\gamma}{2}\theta_2^{\frac{\gamma-1}{2}}\rho_{22}\sigma_{22}\sigma_2 iu_2+\beta_2)C+(\vartheta_2-\frac{1}{2}\sigma_{12}^2)iu_1\\&+(\vartheta_2-\frac{1}{2}\sigma_{22}^2)iu_2+\frac{1}{2}\sigma_{12}^2(iu_1)^2+\frac{1}{2}\sigma_{22}^2(iu_2)^2\\&-\sigma_{12}\sigma_{22}\rho_2u_1u_2=0\end{aligned} \tag{10}$$

$$\begin{aligned}&\frac{\partial A}{\partial t}+\frac{1-\gamma}{2}\sigma_1^2\theta_1^{\gamma}B^2+\frac{1-\gamma}{2}\theta_1^{\frac{1+\gamma}{2}}\rho_{11}\sigma_{11}\sigma_1 iu_1B\\&+\frac{1-\gamma}{2}\theta_1^{\frac{1+\gamma}{2}}\rho_{21}\sigma_{21}\sigma_1 iu_2B+\beta_1\theta_1B+\frac{1-\gamma}{2}\sigma_2^2\theta_2^{\gamma}C^2\\&+\frac{1-\gamma}{2}\theta_2^{\frac{1+\gamma}{2}}\rho_{12}\sigma_{12}\sigma_2 iu_1C+\frac{1-\gamma}{2}\theta_1^{\frac{1+\gamma}{2}}\rho_{22}\sigma_{22}\sigma_2 iu_2C\\&+\beta_2\theta_2C+\vartheta_0(iu_1+iu_2)=0\end{aligned} \tag{11}$$

Using the integral identity:

$\int\frac{dx}{ax^2-bx+c}=\frac{1}{-\sqrt{b^2-4ac}}ln(\frac{2ax-b+\sqrt{b^2-4ac}}{2ax-b-\sqrt{b^2-4ac}})$ we obtain the solutions:

$$\begin{aligned}B(\tau,\mathbf{u})&=\frac{1}{2d_1}\left(f_1+h_1-\frac{2h_1}{1-q_1e^{-h_1\tau}}\right),\\C(\tau,\mathbf{u})&=\frac{1}{2d_2}\left(f_2+h_2-\frac{2h_2}{1-q_2e^{-h_2\tau}}\right).\end{aligned}$$

Integrating both sides of (11) yields:

$$\begin{aligned}A(\tau,\boldsymbol{u})&=d_3\int_0^\tau B(s,\boldsymbol{u})^2ds+f_3\int_0^\tau B(s,\boldsymbol{u})ds\\&+d_4\int_0^\tau C(s,\boldsymbol{u})^2ds+f_4\int_0^\tau C(s,\boldsymbol{u})ds+g_3\int_0^\tau d\,s\\&=\frac{d_3}{4d_1^2}[(f_1+h_1)^2\tau-4h_1\tau(f_1+h_1)\\&-4(f_1+h_1)\,ln\frac{1-q_1e^{-h_1\tau}}{1-q_1}+4h_1^2\tau\\&+4h_1\,ln\frac{1-q_1e^{-h_1\tau}}{1-q_1}+\frac{4h_1}{1-q_1}-\frac{4h_1}{1-q_1e^{-h_1\tau}}]\\&+\frac{f_3\tau}{2d_1}(f_1-h_1)-\frac{f_3}{d_1}ln\frac{1-q_1e^{-h_1\tau}}{1-q_1}\\&+\frac{d_4}{4d_2^2}[(f_2+h_2)^2\tau-4h_2\tau(f_2+h_2)\\&-4(f_2+h_2)\,ln\frac{1-q_2e^{-h_2\tau}}{1-q_2}+4h_2^2\tau\\&+4h_2\,ln\frac{1-q_2e^{-h_2\tau}}{1-q_2}+\frac{4h_2}{1-q_2}-\frac{4h_2}{1-q_2e^{-h_2\tau}}]\\&+\frac{f_4\tau}{2d_2}(f_2-h_2)-\frac{f_4}{d_2}ln\frac{1-q_2e^{-h_2\tau}}{1-q_2}+g_3\tau\end{aligned}$$

4. Digital Power Exchange Option Pricing

This section derives the pricing formula for digital power exchange options. Let $C(t,\alpha_1,\alpha_2,k,k_1,k_2,T)$ denote the price at time t of a European digital power exchange option with maturity T and exchange ratio K The terminal payoff function of the digital power exchange option is defined as:

$$V(\alpha_1,\alpha_2,k,k_1,k_2,T)=\begin{cases}(S_{1T}^{\alpha_1}-KS_{2T}^{\alpha_2})^+\mathbf{1}_{\left(K_1\le\frac{S_{1T}^{\alpha_1}}{S_{2T}^{\alpha_2}}\le K_2\right)}\\\qquad 0\le K\le K_1\\(S_{1T}^{\alpha_1}-KS_{2T}^{\alpha_2})^+\mathbf{1}_{\left(K\le\frac{S_{1T}^{\alpha_1}}{S_{2T}^{\alpha_2}}\le K_2\right)}\\\qquad K_1\le K\le K_2\\\qquad 0,K\ge K_2\end{cases} \tag{12}$$

Where [K1, K2] represents the option's strike interval. Let $k_j=ln\,K_j$. Based on the payoff structure: When $\alpha_1=\alpha_2=1$, the option reduces to a digital exchange option. When $K_1\to-\infty$, and $K_2\to+\infty$, the option becomes a power exchange option.

Theorem 2: Under the market model (1) the time-t price of a European digital power exchange option with maturity T and payoff function (12) is given by:

$$C(t,\alpha_1,\alpha_2,k,k_1,k_2,T)=$$
$$\begin{cases}\psi(t,-i\alpha_1,0,0,0,T)\times\Pi_1(t,k_1,k_2,T)\\-K\times\psi(t,0,-i\alpha_2,0,0,T)\times\Pi_2(t,k_1,k_2,T), & 0\le K\le K_1\\\psi(t,-i\alpha_1,0,0,0,T)\times\Pi_1(t,k,k_2,T)\\-K\times\psi(t,0,-i\alpha_2,0,0,T)\times\Pi_2(t,k,k_2,T), & K_1\le K\le K_2\\0, & K\ge K_2\end{cases}$$

where the coefficient functions satisfy the system:

$$\begin{aligned}\Pi_l(t,k_1,k_2,T)&=\frac{1}{\pi}\int_0^\infty\Re\left[\frac{\phi_l(t,u,T)\times e^{-iuk_1}}{iu}-\frac{\phi_l(t,u,T)\times e^{-iuk_2}}{iu}\right]du,\\\phi_1(t,u,T)&=\frac{\psi(t,(u-i)\alpha_1,-u\alpha_2,0,0,T)}{\psi(t,-i\alpha_1,0,0,0,T)},\\\phi_2(t,u,T)&=\frac{\psi(t,u\alpha_1,-(i+u)\alpha_2,0,0,T)}{\psi(t,0,-i\alpha_2,0,0,T)}.\end{aligned}$$

Proof: Let $Y_t=\alpha_1X_{1t}-\alpha_2X_{2t}$. By risk-neutral pricing, when $0\le K\le K_1$, we have:

$$\begin{aligned}&C(t,\alpha_1,\alpha_2,k,k_1,k_2,T)\\&=E_t^Q[e^{-\int_t^T R_sds}(S_{1T}^{\alpha_1}-KS_{2T}^{\alpha_2})^+1_{\left(k_1\le\frac{S_{1T}^{\alpha_1}}{S_{2T}^{\alpha_2}}\le k_2\right)}]\\&=E_t^Q\left[e^{-\int_t^T R_sds+\alpha_1X_{1T}}1_{(k_1\le Y_T\le k_2)}\right]\\&\quad-KE_t^Q\left[e^{-\int_t^T R_sds+\alpha_2X_{2T}}1_{(k_1\le Y_T\le k_2)}\right]\\&=I_1-KI_2.\end{aligned} \tag{13}$$

To simplify calculations, we perform the following measure transformation:

$$\frac{dQ_l}{dQ}|_{\mathscr{F}_T}=\frac{e^{-\int_t^T R_sds+\alpha_lX_{lT}}}{E_t^Q\left[e^{-\int_t^T R_sds+\alpha_lX_{lT}}\right]},\quad l=1,2$$

It is straightforward to verify that the Radon-Nikodym derivative for the aforementioned measure transformation exists, with $\frac{dQ_l}{dQ}|_{\mathscr{F}_T} = 1$. By Girsanov's theorem, the measures Q1 and Q2 are equivalent martingale measures under Q. Consequently

$$\begin{aligned} I_1 &= E_t^Q\left[e^{-\int_t^T R_s ds+\alpha_1 X_{1T}}\right] Q_1(k_1 \le Y_T \le k_2) \\ &= \psi(t, -i\alpha_1, 0,0,0, T) \Pi_1(t, k_1, k_2, T) \end{aligned} \quad (14)$$

$$\begin{aligned} I_2 &= E_t^Q\left[e^{-\int_t^T R_s ds+\alpha_2 X_{2T}}\right] Q_2(k_1 \le Y_T \le k_2) \\ &= \psi(t, 0, -i\alpha_2, 0,0, T) \Pi_2(t, k_1, k_2, T) \end{aligned} \quad (15)$$

By the uniqueness theorem relating characteristic functions and distribution functions, along with Fourier inversion, we derive

$$\begin{aligned} &\Pi_l(t, k_1, k_2, T) \\ &= \frac{1}{\pi}\int_0^\infty \Re\left[\frac{\phi_l(t,u,T)(e^{-iuk_1}-e^{-iuk_2})}{iu}\right] du, l = 1,2 \end{aligned} \quad (16)$$

where $\phi_l(t, u, T)$ denotes the characteristic function of YT under the measure Q1 defined as

$$\begin{aligned} \phi_1(t, u, T) &= E_t^{Q_1}[e^{iuY_T}] = E_t^Q\left[\frac{e^{-\int_t^T R_s ds+\alpha_1 X_{1T}+iuY_T}}{E_t^Q\left[e^{-\int_t^T R_s ds+\alpha_1 X_{1T}}\right]}\right] \\ &= \frac{E_t^Q\left[e^{-\int_t^T R_s ds+(iu+1)\alpha_1 X_{1T}-iu\alpha_2 X_{2T}}\right]}{E_t^Q\left[e^{-\int_t^T R_s ds+\alpha_1 X_{1T}}\right]} \\ &= \frac{\Psi(t,(u-i)\alpha_1,-u\alpha_2,0,0,T)}{\Psi(t,-i\alpha_1,0,0,0,T)} = \phi_1(t, u). \end{aligned} \quad (17)$$

and

$$\begin{aligned} \phi_2(t, u, T) &= E_t^{Q_2}[e^{iuY_T}] = E_t^Q\left[\frac{e^{-\int_t^T R_s ds+\alpha_2 X_{2T}+iuY_T}}{E_t^Q\left[e^{-\int_t^T R_s ds+\alpha_2 X_{2T}}\right]}\right] \\ &= \frac{E_t^Q\left[e^{-\int_t^T R_s ds+iu\alpha_1 X_{1T}+(1-iu)\alpha_2 X_{2T}}\right]}{E_t^Q\left[e^{-\int_t^T R_s ds+\alpha_2 X_{2T}}\right]} \\ &= \frac{\Psi(t,u\alpha_1,-(i+u)\alpha_2,0,0,T)}{\Psi(t,0,-i\alpha_2,0,0,T)} = \phi_2(t, u). \end{aligned} \quad (18)$$

Similarly, the time-t expression of the option can be derived for the case $K_1 \le K \le K_2$. When $K_2 < K$, the inequality $\frac{S_{1T}^{\alpha_1}}{S_{2T}^{\alpha_2}} \le K_2 < K$ holds, rendering the option value zero.

5. Conclusion

This paper investigates the pricing of digital power exchange options under a non-affine stochastic volatility model. The approximate characteristic function of the log-price distribution of the underlying asset is first derived using the perturbation analysis method for partial differential equations. Then, by applying Fourier transform and its inverse transform, an analytical expression for the digital power exchange options is obtained.

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