DOI: 10.53469/jgebf.2025.07(05).05

Pricing of Basket CDS with Stochastic Interest Rates

Jiaxing Wu

School of Mathematics and Statistics, Guangxi Normal University, Guilin, Guangxi, China

Abstract: ***This paper investigates the pricing of basket credit default swaps (CDS) under stochastic interest rates using a reduced-form model. We assume the default intensity of reference entities and stochastic interest rates both follow Vasicek processes, with risk-free counterparties. Through PDE and ODE methods, we derive approximate closed-form solutions for the joint survival probability density and the probability density of first-default events among reference entities.***

Keywords: Credit risk, Stochastic interest rates, Basket, Credit default swap**.**

1. Introduction

Amidst market volatility and global uncertainties, there has been growing attention on the default risks of major corporations. This has led to increased investor interest in credit derivatives, with credit default swaps (CDS) being particularly prominent.

CDS contracts are of two types: single - name CDS and basket CDS. With globalization accelerating and corporate collaborations intensifying, basket CDS pricing has become a research focus. Current pricing methodologies mainly include:

The structural model pioneered by Black, Scholes, and the reduced-form model proposed by Jarrow and Turnbull Structural models often struggle to obtain explicit solutions for diffusion risk problems, where as reduced-form models demonstrate superior efficacy in such contexts. In reduced-form frameworks, default times are modeled as jump processes of Poisson processes, as demonstrated by Jarro [1]. Lando[2] extended Jarrow's model by employing affine structures to derive closed-form solutions.

Contemporary research favors reduced-form models to avoid data collection complexities. Malherbe [3]represented default intensity via Poisson processes, while Herbertsson and Rootzén [4] developed a novel model addressing multi-credit-risk CDS pricing. Recent advances include: T. Wang and J. Liang[5]extending CDS pricing to fractional Brownian motion environments. Yu Chen and Yu Xing [6] deriving approximate solutions under CEV processes and extending to Vasicek processes Qi Han and Meng Wang [7] establishing CDS pricing formulas using contagion models based on basket CDS properties Yu Xing, Wei Wang, and Xiaonan Su [8]modeling single-name CDS pricing with Hawkes processes while considering counterparty risks.

This study innovatively incorporates stochastic interest rates into basket CDS pricing under a reduced-form Vasicek process framework. This is a critical consideration amid interest rate fluctuations during global market realignments.

The paper is organized as follows:

Section 1 introduces CDS background and model selection rationale.

Section 2 establishes the model framework with Vasicek - process assumptions.

Section 3 derives probability densities via PDE methods.

Section 4 develops pricing formulas under no-arbitrage principles.

Section 5 presents conclusions and future research directions.

2. Model Assumptions

This section establishes the fundamental hypotheses of our model. Traditional credit default swap (CDS) pricing typically considers only single-reference assets under constant interest rate assumptions. However, most contemporary contracts involve basket CDS pricing, and evolving global dynamics—via economic and policy changes—continuously influence interest rates. We therefore propose a basket CDS pricing model incorporating stochastic interest rates.

Assumption 1. When no defaults occur among the reference entities in the basket, the protection buyer F_A is required to pay premiums to the counterparty F_C. Consider a finite time horizon $T > 0$ and a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where:

- $\mathbb{P}$ represents the risk-neutral probability measure;
- $\{\mathcal{F}_t\}_{0\le t\le T}$ denotes the canonical filtration generated by the underlying stochastic structure;
- $\mathbb{P}_t$ indicates the probability measure restricted to $\mathcal{F}_t$;
- the conditional expectation operator is denoted by $\mathbb{E}_t[\cdot]$.

Let τ be a stopping time with respect to the filtration $\{\mathcal{F}_t\}$, satisfying $\tau \le T$. For any sufficiently small $\Delta t \ge 0$, according to the reduced-form model, if the intensity process $\lambda(t)$ exists, then the following holds:

$$\mathbb{P}(\, t < \tau \le t + \Delta t \mid \tau > t\,) = \lambda(t)\Delta t + o(\Delta t). \quad (1)$$

The reference asset consists of a basket of components issued by distinct market entities, denoted as $\{F_{B_i},\, i = 1, \dots, n\}$, entity's default intensity is modeled by $\{\lambda_i(t),\, i = 1, \dots, n\}$. Default under the assumption of risk-free counterparty. Both the default intensities $\{\lambda_i(t),\, i = 1, \dots, n\}$ and the stochastic interest rate $r(t)$ follows Vasicek processes, where the interest rate process $r = \{r(t), t \geq 0\}$ is specified by:

$$d\lambda_i = b_i\big(c_i - \lambda_i(t)\big)dt + \sigma_i dW_i(t), \qquad (2)$$

$$dr(t) = b\big(c - r(t)\big)dt + \sigma dW(t). \qquad (3)$$

Here, b_i, c_i, σ_i and b, c, σ are all non-negative parameters, where: c_i and c $r$epresent the long-term mean levels of $\lambda_i(t)$ and $r(t)$ respectively, b_i and b denote the mean-reversion rates to these long-term levels, $W_i(t)$, $W(t)$ are standard Brownian motions with the following correlation structure: For $i \neq j$, $dW_i(t)dW_j(t) = 0$; for $i = j$, $dW_i(t)dW_j(t) = 1$.

3. Default Probability Density

In this section, we derive the joint survival probability density and the first-to-default probability density for the reference assets. These serve as the foundation for the pricing framework in Section 4.

Theorem 1 *(Joint Survival Probability Density). Under the condition that none of the constituent firms default, we obtain the following partial differential equation (PDE):*

$$\begin{cases} \frac{\partial \hat{p}}{\partial t} + \sum_{i=1}^{n} b_i\,(c_i - \lambda_i)\,\frac{\partial \hat{p}}{\partial \lambda_i} + \frac{1}{2}\sum_{i=j}^{n} \sigma_i^2\,\frac{\partial^2 \hat{p}}{\partial \lambda_j\,\partial \lambda_i} + \\ b(c - r)\,\frac{\partial \hat{p}}{\partial r} - \sum_{i=1}^{n}(\lambda_i + r)\hat{p} = 0, \\ p(s, \lambda; s) = 1. \end{cases} \qquad (4)$$

Proof. Under the survival condition that none of the constituent firms default within the time interval $s(t \leq s \leq T)$, we obtain the survival probability as follows:

$$P_T\{\tau_1 > s, \dots, \tau_n > s\} = \exp\big\{-\int_t^s \sum_{i=1}^{n} \lambda_i\,(u)\mathrm{d}u\big\}. \quad (5)$$

Moreover, since $\mathcal{F}_t \subset \mathcal{F}_T$,by the tower property of conditional expectations, we have: $E_t(1_{\{\text{Event}\}}) = E_t(E_T(1_{\{\text{Event}\}}))$. Consequently, the resulting probability density is given by:

$$\begin{aligned} \hat{p}(t, \lambda, r; s) &= p_t\{\tau_1 > s, \dots, \tau_n > s\} \\ &= E\big[\exp\big\{-\int_t^s \sum_{i=1}^{n} \lambda_i\,(u)\mathrm{d}u\big\}\big|\mathcal{F}_t\big] \\ &= E_t\big[\exp\big\{-\int_t^s \sum_{i=1}^{n} \lambda_i\,(u)\mathrm{d}u\big\}\big]. \end{aligned} \qquad (6)$$

Applying the Feynman-Kac formula, we derive the PDE given in (4).

Theorem 2 *(Semi-Analytical Formula for Joint Survival Probability Density). For the partial differential equation (PDE) (4), we obtain its semi-analytical solution through the following expression:*

$$\hat{p}(t, \lambda; s) = \exp\{A(t, s) - \sum_{i=1}^{n} B_i\,(t, s)\lambda_i(t) - C(t, s)\mathrm{r(t)}\}. \qquad (7)$$

This formulation satisfies:

$$\begin{aligned} B_{i(t,s)} &= \frac{1}{b_i}(1 - e^{-b_i(s-t)}) \\ C_{(t,s)} &= \frac{1}{b}(1 - e^{-b(s-t)}) \\ A(t, s) &= \int_t^s [\tfrac{1}{2}\sum_{i=j}^{n} \sigma_i^2\,\frac{\partial^2 \hat{p}}{\partial \lambda_j\,\partial \lambda_i} - \sum_{i=1}^{n} b_i\,c_i B_i(u, s) \\ &\quad - \sum_{i=1}^{n} b\,cC(u, s)]du. \end{aligned} \qquad (8)$$

Proof. Following Øksendal's[9] approach, the solution for $\hat{p}(t, \lambda, s)$ admits the following form. By substituting the preceding equations into PDE (4), we derive three distinct ordinary differential equations (ODEs):

$$\hat{p}(t, \lambda; s) = \exp\{A(t, s) - \sum_{i=1}^{n} B_i\,(t, s)\lambda_i(t) - C(t, s)r(t)\}, \qquad (9)$$

$$\begin{cases} \frac{\partial A(t,s)}{\partial t} + \frac{1}{2}\sum_{i=j}^{n} \sigma_i^2\,\frac{\partial^2 \hat{p}}{\partial \lambda_j\,\partial \lambda_i} \\ -\sum_{i=1}^{n} b_i\,c_i B_i(t, s) - bcC(t, s) = 0, \\ A(s, s) = 0, \\ \frac{\partial B_i(t,s)}{\partial t} - b_i B_i(t, s) + 1 = 0, \\ B_i(t, s) = 0, \\ \frac{\partial C(t,s)}{\partial t} - bC(t, s) + 1 = 0, \\ C_i(t, s) = 0. \end{cases} \qquad (10)$$

First, solve for $B_i(t, s)$. Because

$$\frac{\partial B_i(t,s)}{\partial t} - b_i B_i(t, s) = -1, \qquad (11)$$

this is a first-order linear ordinary differential equation, for which we can determine its integrating factor as $\mu(t) = e^{-\int_0^t b_i dt} = e^{-b_i t}$, by substituting the integrating factor into Equation (11), we obtain $B_{i(t,s)}$. Similarly, we solve for $C_{(t,s)}$, and then proceed to solve the previous ODEs (9) to determine $A(t, s)$.

Theorem 3. *Suppose the contract terminates when the first of n reference firms defaults, with the seller compensating the buyer. Let s denote this default time s. We define $\hat{p}_i(t, \lambda; s)$ as the probability density of the first firm's default at time s:*

$$\begin{cases} \frac{\partial \hat{p}}{\partial t} + \frac{1}{2}\sum_{i=j}^{n} \sigma_i^2\,\frac{\partial^2 \hat{p}}{\partial \lambda_j\,\partial \lambda_i} + \sum_{i=1}^{n} b_i\,(c_i - \lambda_i)\,\frac{\partial \hat{p}}{\partial \lambda_i} \\ +b(c - r)\,\frac{\partial \hat{p}}{\partial r} - (\sum_{i=1}^{n} \lambda_i + r)\hat{p} = 0. \\ p_i(s, \lambda, s) = \lambda_i. \end{cases} \qquad (12)$$

Proof. By the tower property (or iterated expectation law) of conditional expectations, we have: $E_t(1_{\{\text{Event}\}}) = E_t(E_T(1_{\{\text{Event}\}}))$. The probability density of the first firm defaulting at time $s(t \leq s < \tau_i \leq s + ds \leq T)$ is given by:

$$\begin{aligned} \hat{p}_i(t, \lambda; s) &= p_t\{\tau_1 > s, \dots, \tau_n > s, \tau_i \leq s + \mathrm{d}s\} \\ &= p_t\{\tau_1 > s, \dots, \tau_n > s\}\lambda_i(s)\mathrm{d}s \\ &= E\big[\exp\big\{-\int_t^s \sum_{i=1}^{n} \lambda_i\,(u)\mathrm{d}u\big\}\lambda_i(s)\,\big|\,\mathcal{F}_t\big] \\ &= E_t\big[\exp\big\{-\int_t^s \sum_{i=1}^{n} \lambda_i\,(u)\mathrm{d}u\big\}\lambda_i(s)\big]. \end{aligned} \qquad (13)$$

By applying the Feynman-Kac formula, we derive the partial differential equation (PDE) (12).

Theorem 4. *Under the assumption that the first company defaults at time $\tau_i(s \leq \tau_i \leq s + ds)$,the closed-form solution for $\hat{p}_i(t, \lambda; s)$, representing the probability density of default at time s, is given by:*

$$\hat{p}_i(t, \lambda, s) = \big(D_i(t, s)\lambda_i(\mathrm{t}) + E(t, s)r(t) + H(t, s)\big)\,exp\{A(t, s) - \sum_{i=1}^{n} B_i\,(t, s)\lambda_i(t) - C(t, s)r(t)\}. \qquad (14)$$

At this point,

$$D_i(t, s) = e^{-bi}(s - t),$$

$$E_i(t,s) = 0,$$

$$H(t,s) = \int_t^s \Big[-\sum_{i=1}^n b_i\, C_i B_i(u,s) D_i(u,s) + \frac{1}{2}\sum_{i=1}^n \sigma_i^2\, B_i^2(u,s) D_i(u,s) \Big]\, du. \tag{15}$$

Proof. According to Øksendal, the probability density function $p_i(t,\lambda,r;s)$admits the following analytical form:

$$\hat{p}_i(t,\lambda,r;s) = \big(D_i(t,s)\lambda_i + E(t,s)r(t) + H(t,s)\big)\, exp\{A(t;s) - \sum_{i=1}^n B_i\,(t,s)\lambda_i(t) - C(t,s)r(t)\}. \tag{16}$$

Substituting this expression into PDE (12) yields:

$$\begin{aligned} &\frac{\partial D_i(t,s)}{\partial t}\lambda_i + \frac{\partial E(t,s)}{\partial t} r + \frac{\partial H(t,s)}{\partial t} \\ &+(D_i(t,s)\lambda_i + E(t,s)r + H(t,s)) \\ &\times \Big\{ \frac{\partial A(t,s)}{\partial t} - \sum_{i=1}^n \frac{\partial B_i(t,s)}{\partial t}\lambda_i - \frac{\partial C(s,t)}{\partial t} r \\ &- \sum_{i=1}^n b_i\,(c_i - \lambda_i) B_i(t,s) - b(c-r)C(t,s) \\ &+ \frac{1}{2}\sum_{i=1}^n B_i^2\,(t,s)\sigma_i^2 \lambda_i - \sum_{i=1}^n \lambda_i + r - 1 \Big\} \\ &-b_i(c_i - \lambda_i) D_i(t,s) - b(c-r)E(t,s) \\ &-B_i(t,s) D_i(t,s) \sigma_i^2 \lambda_i \\ &= 0. \end{aligned} \tag{17}$$

From the above equation(s), we can derive the following three ordinary differential equations (ODEs):

$$\begin{cases} \frac{\partial D_i(t,s)}{\partial t} - biD_i(t,s) - 1 = 0, D_i(s,s) = 0 \\ \frac{\partial E(t,s)}{\partial t} - bE(t,s) = 0, E(s,s) = 0 \\ \frac{\partial H(t,s)}{\partial t} - \sum_{i=1}^n b\, ic_i B_i(t,s) D_i(t,s) + \frac{1}{2}\sum_{i=1}^n \sigma_i^2\, B_i^2(t,s) D_i(t,s) = 0, \\ H(s,s) = 0 \end{cases} \tag{18}$$

Solving the above equations yields equation (15).

4. CDS Pricing

In this section, we discuss the pricing of basket CDS under stochastic interest rates. Consider company F_A, holding n bonds issued by companies $\{F_{B_i},\ i = 1, \dots, n\}$. At initial time T, to hedge against default risks from these reference entities, F_A enters into a CDS contract with counterparty F_C, with maturity T. During $[0,T]$, F_A must continuously pay premiums to F_Cs long as none of the $n$reference entities defaults. If any default occurs before T, F_C must compensate F_A. For simplicity, we assume F_C is default-free. As the contract takes effect, the following two scenarios may occur:

Scenario 1: None of the reference entities defaults prior to maturity T.

Scenario 2: A default occurs at time $\tau_i(s \le \tau_i \le s + ds)$, triggering contract termination. Let W denote the premium payments W from F_A to F_C.

For Scenario 1: According to Theorem 3.2, the premiums received by F_C are given by:

$$W \int_t^T e^{-\int_t^s r(u)du}\, \hat{p}(t,\lambda,s)ds. \tag{19}$$

Since none of the companies default by time T, F_A is obligated to continue making premium payments W to F_C throughout the entire contract period $[0,T]$.

Scenario 2: According to Formula (4), the premiums received by F_C are given by:

$$W \int_t^T e^{-\int_t^s r(u)du}\, \hat{p}_i(t,\lambda,s)ds. \tag{20}$$

At this point, F_C must provide compensation to F_A, assuming a recovery rate of R and a total bond face value of L, the compensation payment is calculated as:

$$L(1-R)\int_t^T e^{-\int_t^s r(u)du}\, \hat{p}_i(t,\lambda,s)ds. \tag{21}$$

By the no-arbitrage principle, the CDS contract value at initiation is zero. Therefore, the present value of the total premiums received by F_C must equal the present value of the compensation payments made by F_C. This leads to the following equation:

$$\begin{aligned} &W \int_t^T e^{-\int_t^s r(u)du}\, \hat{p}(t,\lambda,s)ds \\ &+W \int_t^T e^{-\int_t^s r(u)du}\, \hat{p}_i(t,\lambda,s)ds \\ &= L(1-R)\int_t^T e^{-\int_t^s r(u)du}\, \hat{p}_i(t,\lambda,s)ds. \end{aligned} \tag{22}$$

Therefore, the premium payments made by F_A to F_C are given by

$$W = \frac{L(1-R)\int_t^T e^{-\int_t^S r(u)\mathrm{d}u} \hat{p}_i(t,\lambda,s)\mathrm{d}s}{\sum_{i=1}^n \int_t^T e^{-\int_t^S r(u)\mathrm{d}u} \hat{p}_i(t,\lambda,s)\mathrm{d}s + \int_t^T e^{-\int_t^S r(u)\mathrm{d}s} \hat{p}(t,\lambda,s)\mathrm{d}s}. \tag{23}$$

5. Conclusion

This paper investigates the pricing of basket credit default swaps (CDS) under stochastic interest rates. Using a partial differential equation (PDE) approach, we derive an approximate pricing formula. Our model assumes interest rates follow the Vasicek process, reflecting volatility from global economic conditions and policy impacts.

However, to better capture real-world default event clustering, future research could incorporate the Hawkes process for modeling stochastic interest rates. While this would yield more realistic results, it introduces significant computational challenges. Developing a pricing framework for basket CDS under Hawkes-process-driven stochastic interest rates is a promising future direction.

References

[1] R. A. Jarrow and S. M. Turnbull, “Pricing derivatives on fnancial securities subject to credit risk,” The Journal of Finance, vol. 50, no. 1, pp. 53–85, 1995.

[2] D. Lando, “On cox processes and credit risky securities,” Review of Derivatives Re-search, vol. 2, no. 2-3, pp. 99–120, 1998.

[3] A. Herbertsson and H. Rootzén, “Pricing kth-to-default swaps under default conta-gion: the matrix-analytic approach,” Journal of Computational Finance, vol. 12, no.1, pp. 49–78, 2008.

[4] L. Wu, J. Sun, and X. Mei, “Introducing fuzziness in CDS pricing under a structural model,” Mathematical Problems in Engineering, vol. 2018, Article ID 6363474, 7 pages, 2018.

[5] Hao, R., Liu, Y., & Wang, S. (2014). Pricing Credit Default Swap under Fractional Vasicek Interest Rate Model. Journal of Mathematical Finance, 04(01), 10–20.

[6] Chen, Y., & Xing, Y. (2022). Basket Credit Default Swap Pricing with Two Default-able Counterparties. Discrete Dynamics in Nature and Society, 2022(1), 3844001.

[7] Han, Q., & Wang, M. (2023). Pricing of Basket CDS with Bilateral Default Risk under Vasicek Model with Circular Contagion. Mathematical Problems in Engineering, 2023(1), 8960259.

[8] Xing, Y., Wang, W., & Su, X. (2025). Credit default swap pricing with counterparty risk in a reduced form model with Hawkes process. Communications in Statistics - Theory and Methods, 54(6), 1813–1835.

[9] Øksendal, B. 2003. Stochastic differential equations. Berlin, Heidelberg: Springer.