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Evolution and Significance of Sine Function in Ancient Indian Mathematics

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Abstract: ***This research article delves into the historical and mathematical intricacies of the construction of sine functions, particularly focusing on ancient Indian mathematical concepts and their evolution. It begins by highlighting the significance of sine (jya) as a fundamental trigonometric function in Indian mathematics, alongside its synonyms in Sanskrit. The historical journey of the term "sine" is traced back to ancient Indian mathematical texts, which were later translated into Arabic and then into Latin, ultimately leading to the adoption of the term "sine" in modern mathematics. The article elaborates on the definitions and interpretations of sine (ज्या), cosine (कोटिज्या), versed sine (उत्क्रमज्या), and co-versed sine (कोट्युत्क्रमज्या) in Indian mathematical traditions. It explains how these functions were conceptualized and utilized in astronomical calculations, emphasizing their geometric and trigonometric significance. Furthermore, the article discusses the division of a circle into quadrants (Vṛttapādas) and the concept of Trijyā (radius), highlighting the mathematical sophistication and terminology used by ancient Indian mathematicians. Several mathematical formulas related to the construction and calculation of sine functions are presented, including formulas for finding sine values at specific angles such as 45 degrees, 18 degrees and 72 degrees. These formulas showcase the mathematical ingenuity and problem-solving skills of ancient Indian mathematicians. Overall, this research article provides a comprehensive overview of the historical development and mathematical foundations of sine functions in ancient Indian mathematics, shedding light on their enduring legacy and contributions to modern trigonometry.***

Keywords: Trigonometry, Sine, Co-Sine, Versed Sine, Co-Versed Sine, Radius

1. Introduction

The theory of constructing the sine function holds great importance in mathematics. In ancient India, the science of calculating the construction of the sine function was called Jyotpatti (ज्योत्पत्ति). Jya (Sine) serves as the fundamental function in Trigonometry, which comprises six functions: Sine, Cosine, Tangent, Cotangent, Secant, and Cosecant. In Sanskrit, these functions are synonymous with ज्या, कोटिज्या, स्पर्शकः, कोटिस्पर्शकः, छेदकः, कोटिच्छेदकः respectively. Additionally ancient Indian mathematics introduced two more functions known as Versed Sine (उत्क्रमज्या) and Co-versed Sine (कोट्युत्क्रमज्या).

History of Sine
During the eighth century CE, when Indian astronomical texts were rendered into Arabic in Baghdad, the Sanskrit term "jīvā" (जीवा) was adopted to represent the Indian sine function, which Arabs were previously unfamiliar with. In Arabic, it was transcribed as "Jib" or "Jyab", despite lacking significance in the Arabic language. The similarity in consonants allowed for the pronunciation "Jayb" or "Jaib", leading subsequent Arab scholars to sometimes substitute it with the Arabic word "jaib", meaning bosom, fold, or pocket (जेब), either knowingly or due to confusion. Consequently, when Arabic texts were later translated into Latin during the 12th century, Latin translators are opted for the corresponding Latin term "sinus", meaning fold, bosom, or bay, as the equivalent of the Arabic word "Jaib". This eventually gave rise to the anglicized form "sine". Hence, an original Indian mathematical term, "jyā" or "jīvā", after traversing through West Asia, Spain, and England over more than a thousand years, ultimately returned to India during British rule as "sine".

Sine (Jyā) –
The Sanskrit word Jya means a bow-string and hencethe chord of an arc. The arc of a circle is called Chapa in Sanskrit. Generaly, Jya means the straight line of one point to another in a circumference of a circle which is known as 'chord' in English. But in astronomical calculations, half of a chord is called jya which is called Sine in English. Bhaskaracharya clearly says in Siddhanta Shiromani –

अर्धज्याग्रे खेचरो मध्यसूत्रात् तिर्यक्संस्थो जायते येन तेन ।
अर्धज्याभिः कर्म सर्वं ग्रहाणामर्धज्यैव ज्याभिधानात्र वेद्या ॥[1]

That means, the plannets moves in their orbits on the tip of ardhajya. Hence all astronomical calculations are in the base on ardhajya. So in Astronomy, ardhajya is called jya.

Jya or sine is the horizontal line drawn from the apex of the arc towards the base of the arc to the diameter.

Cosine (Kotijyā) –
The Sanskrit term "koti" encompasses meanings such as "the curved end of a bow" or "the end or extremity in general." In the realm of Trigonometry, it evolved to represent "the complement of an arc of 90°". Therefore, the fundamental meaning of the term "koti-jya" or cosine is essentially the sine of the complementary arc.

Versed Sine (Utkramajyā or Bhujotkramajyā) –
Utkruma refers to a concept of reversal, outward movement, or surpassing. Therefore, the term utkramajya essentially

[1] Śiddhāntaśiromaniḥ – Grahaganitādhyāyaḥ – Spaśtādhikāraḥ – Verse 2

translates to "reversed sine." This function earns its name in contrast to krama-jya because its tabulated values are obtained by subtracting elements from the radius in reverse order from the tabulated values of the latter. In simpler terms, it represents the exceeding part of the krama-jya, considered in reverse sequence.Thus, in Sürya Siddhanta it is stated:

प्रोज्झ्योत्क्रमेण व्यासार्धादुत्क्रमज्यार्धपिण्डकाः ॥[2]

"The (tabular) versed sines are obtained by subtracting from the radius the (tabular) sines in the reversed order."
Radius – Cosineθ = Versed Sine

Versed Co-Sine (Kotyutkramajyā) –
The segment of the radius from the point to which the tip of the cosine touches the radius to the arc is called co-versed sine.
Radius – Sineθ = Co-versed Sine

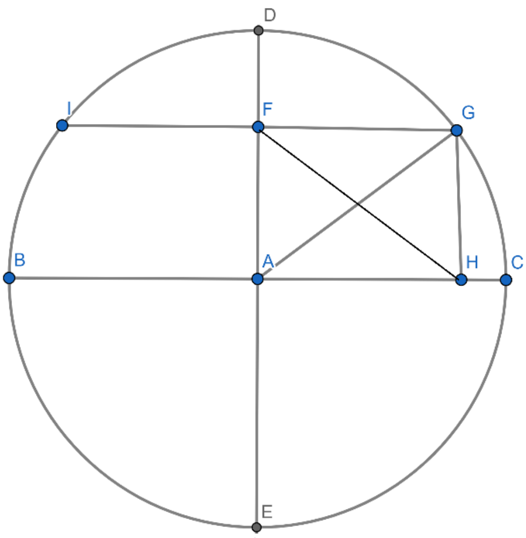

Figure 1: Sine functions in a circle

In Figure 1,
$\widehat{DG}$ = Arc
IG = Chord
FG = Sine
GH = Cosine
FD = Versed Sine
HC = Co-versed Sine
AD = AG = AC = Radius
AFGH Parallelogram. Opposite bases of parallelogram are equal.
Hence, HF = AG

$$\sqrt{FG^2 + GH^2} = HF = AG$$
$$\sqrt{AG^2 - GH^2} = FG$$
$$\sqrt{AG^2 - FG^2} = GH$$

AD – GH = AD – AF = FD = Versed Sine
AC – FG = AC – AH = HC= Co-versed Sine

Quardants (Vṛttapādas):
A circle is commonly divided into four parts called quadrants by two perpendicular lines, typically the east-to-west line and the north-to-south line. These quadrants are further categorized into odd (ayugma or vishama) and even (yugma or sama).

Each quadrant comprises 90 degrees or 3 signs. Each sign encompasses 30 degrees, and each degree consists of 60 arc-minutes. Moreover, each arc-minute contains 60 arc-seconds.

Table 1 : Angular Measurement System in Ancient Indian Mathematics

1 Circle	12 Zodiac Signs / 360 degrees / 4 quardants
1 quardant	3 Zodiac Signs / 90 degrees
1 Sign	30 degree
1 degree	60 arc-minutes
1 arc-minute	60 arc-seconds

Trijyā (Radius)
The evolution of the term "trijya" for vyāsārdha after the introduction of the concept of jyā is intriguing and merits special attention. In line with astronomical conventions of dividing the sun's ecliptic into 12 rashis (zodiacal signs), the circumference of any circle was similarly divided into 12 equal parts, each termed a rāśi. Consequently, a sign or rashi represented an angular measure of 30 degrees, and a quadrant of a circle encompassed three signs.

With the advent of the definition of jya as the Indian sine of an arc in a circle, which represents half the chord of twice the arc, the jyā of a quadrant was realized to be equivalent to its radius. In essence, when considering a quadrant, twice its arc amounts to a semicircle, and its chord becomes the diameter, half of which is the radius. Hence, the jyā of a quadrant translates to the radius. Consequently, the radius of the reference circle came to be known as trijya or "tri-sine" denoting the sine of three signs (tri-rāśi-jya), which was eventually shortened for convenience.

This transition in terminology also extended to other designations such as tri-jīvā, etc. However, the technical distinction between trijya (sign of three signs) or the Sinus Totus (total or complete sine) and vyāsārdha (semi-diameter) remained recognized whenever necessary.

Moreover, various values were employed for the Sinus Totus by Indians over the ages, including 43, 60, 120, 150, 200, 300, 500, 1000, 3270, 3438 and 3600, alongside typically Indian values rooted in specific relations.

Additionally, the radius has been referred to by other terms such as viskambhārdha, vyāsārdha etc, all of which are ancient designations. This demonstrates how trigonometry not only introduced novel mathematical tools but also enriched the lexicon of mathematical terminology over time.

$$R = \left(\frac{21600}{2\pi}\right) minutes.$$

Some basic formulas regarding construction of sine:
Below are discussed some basic formulas for constructing sine as per ancient Indian mathematics.

[2] Sūryasiddhāntaḥ – Spaśtādhikāraḥ – Verse 22

Formula to find sin 45° –

Bhaskaracharya states the formula to find sin 45° in Goladhyaya of Siddhanta Siromani as follows:

त्रिभमौर्विकायाः वर्गार्धमूलं शरवेदभागजीवा ।[3]

$$\sqrt{\frac{R^2}{2}} = sin45°$$

In this formula, the radius is represented by 'R'. This formula represents the relationship between the square root of half the square of the radius of a circle and the sine of 45 degrees. It states that the square root of half the square of the radius of a circle is equal to the sine of 45 degrees. In simpler terms, it means that if you take the radius of a circle, square it, divide it by 2, and then find the square root of that value, you will get the sine of 45 degrees. This relationship arises from the geometry of a circle and the properties of trigonometric functions.

In modern trigonometry, the radius is often taken as 1. So, when the radius is 1, this formula becomes as follows:

$$\sqrt{\frac{1}{2}} = sin45°$$

Two Formulas to find sin$\frac{\theta}{2}$ –

Bhaskaracharya also states two formulas to find sin$\frac{\theta}{2}$. The first formula is:

क्रमोत्क्रमज्याकृतियोगमूलाद्दलं तदर्धांशकशिञ्जिनी स्यात् ।[4]

$$\frac{\sqrt{sin^2\theta + versine^2\theta}}{2} = \sin\frac{\theta}{2}$$

This formula illustrates a trigonometric identity relating the sine (sin) and versine (R – cos) functions. It states that the square root of the sum of the squares of the sine and versine of an angle (θ), divided by 2, equals the sine of half that angle ($\frac{\theta}{2}$). In simpler terms, if you take the sine and versine of an angle, square them, add the results, find the square root of that sum, and then divide by 2, you will obtain the sine of half the angle. This identity is a consequence of trigonometric properties and is often used in trigonometric calculations and proofs.

The second formula is:

त्रिज्योत्क्रमज्यानिहतेर्दलस्य मूलं तदर्धांशकशिञ्जिनी वा ।[5]

To find sin$\frac{\theta}{2}$, we need to multiply the radius of the circle by the versine of θ, divide by two, and then find the square root.

$$\sqrt{\frac{R \times Versine\ \theta}{2}} = \sin\frac{\theta}{2}$$

This formula represents a relationship between the radius of a circle, the versine of an angle (θ), and the sine of half that angle ($\frac{\theta}{2}$). It states that the square root of the product of the radius and the versine of the angle, divided by 2, equals the sine of half the angle. In simpler terms, if you take the product of the radius and the versine of an angle, find the square root of half that product, you will get the sine of half that angle. This relationship arises from trigonometric properties and is useful in various geometric and trigonometric calculations.

In modern trigonometry, the radius is often taken as 1. So, when the radius is 1, this formula becomes as follows:

$$\sqrt{\frac{Versine\ \theta}{2}} = \sin\frac{\theta}{2}$$

Formula to find sin 18° and sin 72° –

Samrat Jagannatha mentioned a method of finding sine 18° and sine 72° in Siddhanta Samrat. This method is detailed in the following verse:

त्रिज्यार्धवर्गादिषुसङ्गुणाच्च मूलं खरामांशकजीवयोनम् ।
तदर्धकं स्याद्धृतिभागजीवा तत्कोटिजीवा द्विनगांशकानाम्
॥[6]

$$sin\ 18° = \frac{\sqrt{\left(\frac{R}{2}\right)^2 \times 5} - \sin 30°}{2}$$

This formula provides a method for calculating the sine of 18 degrees. It begins by taking the radius of a circle, dividing it by 2, and then squaring the result. Next, this squared value is multiplied by 5. After finding the square root of this product, the sine of 30 degrees is subtracted from it. Finally, the resulting value is divided by 2 to obtain the sine of 18 degrees.

The cosine of 18 degrees is equal to the sine of 72 degrees.
cos 18° = sin 72°

2. Conclusion

In conclusion, this research article provides valuable insights into the historical evolution and mathematical intricacies of sine functions in ancient Indian mathematics. It highlights the significance of sine as a fundamental trigonometric function and discusses its synonyms, along with related functions such as cosine, versed sine, and co-versed sine.

The article traces the journey of the term "sine" from its origins in Sanskrit to its adoption and transformation in Arabic and Latin mathematical traditions, showcasing the cultural and linguistic influences on mathematical terminology.

Furthermore, it explores the geometric interpretations of sine functions, their use in astronomical calculations, and the division of circles into quadrants. The concept of trijya (radius) is also discussed, demonstrating the mathematical sophistication of ancient Indian mathematicians.

Through various mathematical formulas and methods presented in the article, such as those for finding sine values

[3] Siddhāntaśiromaniḥ – Golādhyāyah – Cedyakādhikāraḥ – Verse 3
[4] Siddhāntaśiromaniḥ – Golādhyāyah – Cedyakādhikāraḥ – Verse 4
[5] Siddhāntaśiromaniḥ – Golādhyāyah – Cedyakādhikāraḥ – Verse 5
[6] Śiddhāntasamrāt – Jyotpattiḥ – Verse 11

at specific angles, the article underscores the problem-solving skills and mathematical ingenuity of ancient Indian scholars.

Overall, this research article contributes to a deeper understanding of the historical context and mathematical foundations of sine functions, highlighting the rich heritage and enduring contributions of ancient Indian mathematics to modern trigonometry.

References

[1] Vateśvara Siddhāntaḥ; Author : Vateśvarācārya, Hindi Commentary : Satyadev Sharma, First Edition (2019 CE), Parimala Publications – New Delhi.
[2] Siddhanta Śiromaniḥ (with Vāsanābhāśya of author and Vāsanāvārtikā of śrī Nṛsimhadaivajna); Author : Bhāskarācārya, Edition : 2010 CE, Sampoornananda Sanskrit University – Varanasi.
[3] Siddhāntaśekharaḥ; Author : Ṣrīpati, Hindi Commentary : Satyadev Sharma, Chowkhamba Krishnadas Acadamy – Varanasi.
[4] Siddhāntasamrāt; Author : Śamrāt Jagannātha, First Edition (1976 CE), Sanskrit Parishad – Sagara University – Sagara (Madhya Pradesh)
[5] Sūryasiddhānta; Sanskrit Commentary – Ṣrī Kapileśvara Śastrī, Edition : 2015 CE, Chowkhambha Sanskrit Bhavan – Varanasi.
[6] Saralatrikonamitiḥ; Author : Bapūdeva Śastrī, Edition : 2002 CE, Sampoornananda Sanskrit University – Varanasi.
[7] www.google.com