DOI: 10.53469/jerp.2026.08(02).15

Integration of Multivariable Polynomials with Hypergeometric Functions: Analytical Solutions

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Abstract: ***The purpose of this paper is to establish some certain integrals involving G-function and multivariable polynomial as product. Integrals are obtained by combining the products of first- and second-class general polynomials with the g-function of one variable and n- variables.***

Keywords: Multivariable Hypergeometric function, Fox G-function, Multivariable Polynomial.

1. Introduction:

In this paper we derive some integrals involving the product of a G-function of one variable and a multivariable with multivariate polynomials of first and second class as defined by Srivastava (1985). We will use the following formulae in our current investigation. The G-function of one variable given by Meijer (1936)

$$G_{P,Q}^{M,N}\left(z\left|\begin{matrix}a_1,a_2,...,a_p\\ b_1,b_2,...,b_q\end{matrix}\right.\right)$$

$$=\frac{1}{2\pi i}\int_\gamma \frac{\prod_{j=1}^{M}\Gamma(b_j-\xi)\prod_{j=1}^{N}\Gamma(1-a_j+\xi)}{\prod_{j=M+1}^{Q}\Gamma(1-b_j+\xi)\prod_{j=N+1}^{P}\Gamma(a_j-\xi)}z^\xi d\xi \tag{1}$$

Where $i=\sqrt{-1}$, $z\neq 0$; $z^\xi=\exp\{\log|z|+i\arg z\xi\}$

In which $\log|z|$ represent a natural logarithm of $|z|$ and $i\arg z$ does not necessarily have the principal value.

Where M, N, P and Q are integers with constrained $0\le N\le P$, $0\le M\le Q$, and $a_j[j=1,2,...,p]$, $b_j[j=1,2,...,q]$ are complex numbers such that coincides with any pole of $\Gamma(1-a_j+\xi), j=1,2,...,N$. An empty product is interpreted as 1. This perception will persist everywhere.

(I) The path γ runs from $(\varepsilon-i\infty)$ to $(\varepsilon+i\infty)$ in such a way that all the poles of $\Gamma(b_j-\xi)$, $j=1,2,...,M$ lie on the right side and all the points of $\Gamma(1-a_j+\xi)$, $j=1,2,....,N$ lie on the left side of the path. The integral converges absolutely if $2(M+N)>P+Q$, and $|\arg z|<\frac{\pi}{2}[2(M+N)-P-Q]$.

If $|\arg z|=\frac{\pi}{2}[2(M+N)-P-Q]\ge 0$ the integral converges absolutely when $P=Q$, if $R(\lambda)+1<0$; and when $P\neq Q$, if with $\xi'=\sigma+i\varepsilon$, σ and ε are real, then σ is chosen so that for $\varepsilon\to\pm\infty$, $(Q-P)\sigma>\left[R(\lambda)+1+\left(\frac{P-Q}{2}\right)\right]$,

where $\lambda=\sum_{j=1}^{Q}b_j-\sum_{j=1}^{P}a_j$.

(II) The path γ is the loop from $+\infty$ to $+\infty$ including all poles of $\Gamma(b_j-\xi), j=1,2,...,M$ once in the negative direction, but none of the poles of $\Gamma(1-a_j+\xi), j=1,2,....,N$. If $Q\ge 1$ and either $P<Q$ or $P=Q$, and $|z|<1$ then the integral converges.

(III) The path γ is a loop from $-\infty$ to $-\infty$, including all poles of $\Gamma\left(1-a_j+\xi\right), j=1,2,....,N$, once in the positive direction, but none of the poles of $\Gamma(b_j-\xi), j=1,2,...,M$. If $P\geq 1$ and are either $P=Q$, and $|z|>1$ then the integral converges.

It is further assumed that the values of variables z and parameters are such that at least one of the above definitions make sense.

The G-function of multivariable is given by

$$G^{M,N;m_1,n_1;...;m_r,n_r}_{P,Q;p_1,q_1;...;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left|\begin{matrix}(a_j)_{1,P};(c_1)_{1,p_1}...(c_r)_{1,p_r} \\ (b_j)_{1,Q};(d_1)_{1,q_1}...(d_r)_{1,q_r}\end{matrix}\right.\right] \tag{2}$$

$$=\frac{1}{2\pi i}\int_{\gamma_1}...\int_{\gamma_r}\psi\left(\xi_1,\xi_2,...,\xi_r\right)\prod_{k=1}^{r}\phi_k\left(\xi_k\right)z_k^{\xi_k}d\xi_k$$

$$\psi\left[\xi_1,\xi_2,...,\xi_r\right]=\frac{\prod_{j=1}^{M}\Gamma\left[b_j-\sum_{k=1}^{r}\xi_k\right]\prod_{j=1}^{N}\Gamma\left[1-a_j+\sum_{k=1}^{r}\xi_k\right]}{\prod_{j=M+1}^{Q}\Gamma\left[1-b_j+\sum_{k=1}^{r}\xi_k\right]\prod_{j=N+1}^{P}\Gamma\left[a_j-\sum_{k=1}^{r}\xi_k\right]} \tag{3}$$

And

$$\phi_k\left(\xi_k\right)=\frac{\prod_{j=1}^{m_k}\Gamma\left[d_{kj}-\xi_k\right]\prod_{j=1}^{n_k}\Gamma\left[1-c_{kj}+\xi_k\right]}{\prod_{j=m_k+1}^{q_k}\Gamma\left[1-d_{kj}+\xi_k\right]\prod_{j=n_k+1}^{p_k}\Gamma\left[c_{kj}-\xi_k\right]} \tag{4}$$

And P,Q,M,N,m_k,n_k,p_k,q_k, k are positive integers with constrained $p\geq N\geq 1, 0\leq M\leq Q$ $q_k\geq m_k\geq 0$ and $p_k\geq n_k\geq 0$ $k=1,2,...,r$ The path γ_r situated in the complex plane, which moves from $-i\infty$ to $+i\infty$, such that all poles of $\Gamma\left(d_{kj}-\xi_k\right)$, $j=1,..,m_k$ and $\Gamma\left(1-a_j+\sum_{k=1}^{r}\xi_k\right)$ $j=1,...,N$, are to the left of γ_r.

Srivastava (1985;686) defined the second-class of multivariable polynomial as follows

$$S^{\beta_1,...,\beta_t}_{\alpha_1,...,\alpha_t}\left[z_1,z_2,...,z_t\right]=\sum_{k_1=0}^{\left[\frac{\alpha_1}{\beta_1}\right]}...\sum_{k_t=0}^{\left[\frac{\alpha_t}{\beta_t}\right]}\left(-\alpha_1\right)_{\beta_1k_1}...\left(-\alpha_t\right)_{\beta_tk_t}\times A\left(\alpha_1,k_1;...;\alpha_t,k_t\right)\frac{z_1^{k_1}}{k_1!}...\frac{z_t^{k_t}}{k_t!} \tag{5}$$

Srivastava and Garg (1987:686) defined the first-class of multivariable polynomial as follows

$$S^{\beta_1,\beta_2,...,\beta_t}_{\alpha}\left[z_1,z_2,...,z_t\right]=\sum_{k_1,k_2,...,k_t=0}^{\beta_1k_1+...+\beta_tk_t\leq\alpha}\left(-\alpha\right)_{\beta_1k_1+...+\beta_tk_t}\times A\left(\alpha;k_1;...;k_t\right)\frac{z_1^{k_1}}{k_1!}...\frac{z_t^{k_t}}{k_t!} \tag{6}$$

where $\alpha=0,1,2,...$

From the table of integration [Gradshteyn and Ryzhik (2007):3.196 Eq. 3 and 3.257 eq. 3)] We need the following integration formulas

$$\int_{-1}^{1}\left(1-x\right)^{\rho}\left(1+x\right)^{\sigma}dx=2^{\rho+\sigma+1}B\left(\rho+1,\sigma+1\right) \tag{7}$$

$$\int_{0}^{\infty}\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-\rho-1}dx=\frac{\sqrt{\pi}\,\Gamma\left(\rho+\frac{1}{2}\right)}{2a\left(4ab+c\right)^{\rho+\frac{1}{2}}\Gamma\left(\rho+1\right)} \tag{8}$$

$\mathrm{Re}\left(\rho\right)+\frac{1}{2}>0$

2. Main Result:

2.1 First Integral:

The first integral is obtained from the product of the G-function of one variable and the second-class of multivariate polynomial.

$$\int_{-1}^{1}\left(1-y\right)^{\rho}\left(1+y\right)^{\sigma}S^{\beta_1,...,\beta_t}_{\alpha_1,...,\alpha_t}\left[z_1\left(1-y\right)^{m_1}\left(1+y\right)^{n_1},...\right.$$
$$\left...,z_t\left(1-y\right)^{m_t}\left(1+y\right)^{n_t}\right]G^{M,N}_{P,Q}\left[z\left(1-y\right)\left(1+y\right)\left|\begin{matrix}(a_j)_{1,P}\\(b_j)_{1,Q}\end{matrix}\right.\right]dy$$
$$=2^{\rho+\sigma+1}S^{\beta_1,...,\beta_t}_{\alpha_1,...,\alpha_t}\left[2^{(m_i+n_1)}z_1,...,2^{(m_i+n_t)}z_t\right]$$
$$G^{M,N+2}_{P+2,Q+1}\left[4z\left|\begin{matrix}\left(-\sigma-\sum_{i=1}^{t}n_ik_i\right);\left(-\rho-\sum_{i=1}^{t}m_ik_i\right);\left(a_j\right)_{3,P+2}\\ \left(b_j\right)_{1,Q};\left(-2-\sigma-\rho-\sum_{i=1}^{t}\left(m_i+n_i\right)k_i\right)\end{matrix}\right.\right] \tag{9}$$

Where $m_1,m_2,...,m_t>0,\ n_1,n_2,...,n_t>0$

2.2 Second Integral:

Second integral is obtained from the product of G-function of one variable and first class of multivariable polynomial

$$\int_{-1}^{1}\left(1-y\right)^{\rho}\left(1+y\right)^{\sigma}S^{\beta_1,\beta_2,...,\beta_t}_{\alpha}\left[z_1\left(1-y\right)^{m_1}\left(1+y\right)^{n_1},...,\right.$$
$$\left.z_t\left(1-y\right)^{m_t}\left(1+y\right)^{n_t}\right]G^{M,N}_{P,Q}\left[z\left(1-y\right)\left(1+y\right)\left|\begin{matrix}(a_j)_{1,P}\\(b_j)_{1,Q}\end{matrix}\right.\right]dy$$
$$=2^{\rho+\sigma+1}S^{\beta_1,\beta_2,...,\beta_t}_{\alpha}\left[2^{m_1+n_1}z_1,...,2^{m_t+n_t}z_t\right]$$
$$G^{M,N+2}_{P+2,Q+1}\left[4z\left|\begin{matrix}\left(-\sigma-\sum_{i=1}^{t}n_ik_i\right);\left(-\rho-\sum_{i=1}^{t}m_ik_i\right);\left(a_j\right)_{3,P+2}\\ \left(b_j\right)_{1,Q};\left(-3-\sigma-\rho-\sum_{i=1}^{t}\left(m_i+n_i\right)k_i\right);\end{matrix}\right.\right] \tag{10}$$

Where $m_1,m_2,...,m_t>0,n_1,n_2,...,n_t>0$

2.3 Third Integral:

Third integral is obtained from the product of G-function of multivariable and second-class of multivariable polynomial

$$\int_0^\infty \left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-\rho-1} S_{\alpha_1,\dots,\alpha_t}^{\beta_1,\dots,\beta_t}\left[z_1\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{m_1},\dots, \dots,z_t\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{m_t}\right] G_{P,Q;p_1,q_1;\dots;p_r,q_r}^{M,N;m_1,n_1;\dots;m_r,n_r}\left[y_1\left\{\left(ax+\frac{b}{x}\right)^2+c\right\},\dots, \dots,y_t\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}\middle| \begin{matrix}(a_j)_{1,P};(c_1)_{1,p_1}\dots(c_r)_{1,p_r}\\(b_j)_{1,Q};(d_1)_{1,q_1}\dots(d_r)_{1,q_r}\end{matrix}\right]dx$$

$$=\frac{\sqrt{\pi}}{2a(4ab+c)^{\rho+1/2}} S_{\alpha_1,\dots,\alpha_t}^{\beta_1,\dots,\beta_t}\left[z_1(4ab+c)^{m_1},\dots,z_t(4ab+c)^{m_t}\right] G_{P+1,Q+1;p_1,q_1;\dots;p_r,q_r}^{M,N+1;m_1,n_1;\dots;m_r,n_r}\left[\begin{matrix}y_1(4ab+c)^{-1},\\ \vdots \\ y_t(4ab+c)^{-1}\end{matrix}\middle| \begin{matrix}\left(-\rho+\frac{1}{2}+\sum_{i=1}^t m_ik_i\right);(a_j)_{2,P+1};(c_1)_{1,p_1}\dots(c_r)_{1,p_r}\\(b_j)_{1,Q};\left(-\rho+\sum_{i=1}^t m_ik_i\right);(d_1)_{1,q_1}\dots(d_r)_{1,q_r}\end{matrix}\right] \tag{11}$$

3. Proof

To establish the first integral formula, we use the second class of multivariable polynomial given by equation [5] and the G-function of one variable given on the left side of equation [1] in terms of a contour integral of Mellin–Barnes type. After interchanging the order summation and integration, we get the following relation, then after simplifying it a bit, we get

$$\sum_{k_1=0}^{[\alpha_1/\beta_1]}\dots\sum_{k_t=0}^{[\alpha_t/\beta_t]}(-\alpha_1)_{\beta_1k_1}\dots(-\alpha_t)_{\beta_tk_t} A(\alpha_1,k_1;\dots;\alpha_t,k_t)\frac{z_1^{k_1}}{k_1!}\dots\frac{z_t^{k_t}}{k_t!}\times \frac{1}{2\pi i}\int_c \phi(\xi)z^{\xi}\int_{-1}^{1}(1-y)^{\rho+\xi+\sum_{i=1}^t m_ik_i}(1+y)^{\sigma+\xi+\sum_{i=1}^t n_ik_i}\,dxd\xi$$

Now with the help of integral [7] and interpreting the resulting contour integral of the G-function we find the first integral [9]. Similar to the proof of equation [9], we can also establish the second integral [10].

To obtain the result [11] first we express the G-function of the multivariate [2] as a contour integral of Millen-Barnes type and the first class of polynomial in the series form given on the left side of the equation [5] and integration [8] is written as a product. Now interchanging the order of integration and summation, we get the following relation, then after a little simplification we get

$$\sum_{k_1=0}^{[\alpha_1/\beta_1]}\dots\sum_{k_t=0}^{[\alpha_t/\beta_t]}(-\alpha_1)_{\beta_1k_1}\dots(-\alpha_t)_{\beta_tk_t} A(\alpha_1,k_1;\dots;\alpha_t,k_t)\times \frac{z_1^{k_1}}{k_1!}\dots\frac{z_t^{k_t}}{k_t!}\frac{1}{2\pi i}\int_{\lambda_1}\dots\int_{\lambda_r}\psi(\xi_1,\xi_2,\dots,\xi_r)\prod_{k=1}^{r}\phi_k(\xi_k)z_k^{\xi_k}\times \int_0^\infty\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-1-\rho+\sum_{i=1}^t m_ik_i-\sum_{k=1}^r \xi_k}dxd\xi_k$$

where $\xi_1,\xi_2,\dots,\xi_r$ are the variables of the Mellin–Barnes type contour integral of the G-function as mentioned above [2]. Now with the help of integration [8] and interpreting the resulting contour integral in terms of the G-function of the r-variable we easily get the result [11].

4. Special Cases

If we take $A(\alpha_1,k_1;\dots;\alpha_t,k_t)=A_1(\alpha_1,k_1)\dots A_t(\alpha_t,k_t)$ in equations [9] and [11] then the multivariate polynomial $S_{\alpha_1,\dots,\alpha_t}^{\beta_1,\dots,\beta_t}[z_1,z_2,\dots,z_t]$ turns into $S_{\alpha_1}^{\beta_1}[z_1]\times S_{\alpha_2}^{\beta_2}[z_2]\times\dots\times S_{\alpha_t}^{\beta_t}[z_t]$, a product of polynomials $S_{\alpha}^{\beta}[z]$ defined by [Srivastava, 1972;1, eq.(1)]The result of equation [9} change into

$$\int_{-1}^{1}(1-y)^{\rho}(1+y)^{\sigma}S_{\alpha_1,\dots,\alpha_t}^{\beta_1,\dots,\beta_t}\left[\sigma_X z_1(1-y)^{m_1}(1+y)^{n_1},\dots, \dots,z_1(1-y)^{m_1}(1+y)^{n_1}\right]G_{P,Q}^{M,N}\left[z(1-y)(1+y)\middle|\begin{matrix}(a_j)_{1,P}\\(b_j)_{1,Q}\end{matrix}\right]dy$$
$$=2^{\rho+\sigma+1}\prod_{j=1}^{t}S_{\alpha_j}^{\beta_j}\left[2^{(m_j+n_j)}z_j\right] G_{P+2,Q+1}^{M,N+2}\left[4z\middle|\begin{matrix}\left(-\sigma-\sum_{i=1}^t n_ik_i\right);\left(-\rho-\sum_{i=1}^t m_ik_i\right);(a_j)_{3,P+2}\\(b_j)_{1,Q};\left(-2-\sigma-\rho-\sum_{i=1}^t(m_i+n_i)k_i\right)\end{matrix}\right] \tag{12}$$

And the result of equation [11] change into

$$\int_0^\infty\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-\rho-1}G_{P,Q;p_1,q_1;\dots;p_r,q_r}^{0,N;m_1,n_1;\dots;m_r,n_r}\left[y_1\left\{\left(ax+\frac{b}{x}\right)^2+c\right\},\dots, y_t\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}\middle|\begin{matrix}(a_j)_{1,P};(c_1)_{1,p_1}\dots(c_r)_{1,p_r}\\(b_j)_{1,Q};(d_1)_{1,q_1}\dots(d_r)_{1,q_r}\end{matrix}\right] \prod_{j=1}^{t}S_{\alpha_j}^{\beta_j}\left[z_j\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{m_j}\right]dx$$

$$= \frac{\sqrt{\pi}}{2a(4ab+c)^{\rho+1/2}} \prod_{j=1}^{t} S_{\alpha_j}^{\beta_j}\left[z_j(4ab+c)^{m_j}\right]$$

$$G_{P+1,Q+1;p_1,q_1;\dots;p_r,q_r}^{M,N+1;m_1,n_1;\dots;m_r,n_r}\left[\begin{matrix} y_1(4ab+c)^{-1} \\ \vdots \\ y_t(4ab+c)^{-1}\end{matrix}\middle|\begin{matrix}\left(-\rho+\frac{1}{2}+\sum_{i=1}^{t} m_i k_i\right);(a_j)_{2,P+1};(c_1)_{1,p_1}\dots(c_r)_{1,p_r} \\ (b_j)_{1,Q};\left(-\rho+\sum_{i=1}^{t} m_i k_i\right);(d_1)_{1,q_1}\dots(d_r)_{1,q_r}\end{matrix}\right] \quad (13)$$

If we take $A(\alpha_1,k_1;\dots;\alpha_t,k_t) = \frac{(\delta_1)_{k_1\tau_1+\dots+k_t\tau_t}}{(\gamma_1)_{k_1\lambda_1+\dots+k_t\lambda_t}}$ in equation [9] then the second class of polynomial $S_{\alpha_1,\dots,\alpha_t}^{\beta_1,\dots,\beta_t}[z_1,z_2,\dots,z_t]$ will turn into first-class hypergeometric polynomial of multivariate and when we put it in equation [10] then the first-class polynomial $S_{\alpha}^{\beta_1,\dots,\beta_t}[z_1,z_2,\dots,z_t]$ will turn into second class hypergeometric polynomial of multivariable, defined by Srivastava and Garg, (1987) and we easily obtain two new integrals involving polynomials.

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