

Bridging Theoretical Foundations with Practical Applications in Nonlinear Analysis and Optimization

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Abstract: Z. Mustafa [8] introduced generalized metric space called b2-metric space. Kamran et al. [7], have dealt with an extended b-metric space. The aim of this paper is to establish a notion of a generalized extended b2 metric space which extends and generalizes metric space due to Z. Mustafa [8], Khan et al [10] and Kamran et al. [7]. Also we prove a fixed point theorem on a generalized extended b2 metric space.

Keywords: metric space, b metric space, 2 metric space, generalized 2 metric space, extended 2 metric space.

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1. Introduction

The notion of a 2-metric space was introduced by Gahler, in [4]. Several fixed-point results were obtained in [1,2,3,4,5 6], as a generalization of the concept of a metric space. A 2-metric is not a continuous function of its variables, whereas an ordinary metric is. The basic philosophy is that since a 2-metric measures area, a contraction should send the space towards a configuration of zero area, which is to say a line.

Z. Mustafa introduced a new type of generalized metric space called b2-metric space, as a generalization of the 2-metric space, [8].

Recently, Kamran et al., have dealt with an extended b-metric space and obtained unique fixed-point results, [7].

Definition 1.1. [4, 9] Let X be a non-empty set and $d : X \times X \times X \rightarrow \mathbb{R}_+$ be a map satisfying the following properties

(i) $d(x,y,z) = 0$ if at least two of the three points are the same .

(ii) For $x, y \in X$ such that $x \neq y$ there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.

(iii) symmetry property: for $x, y, z \in X$,

$$d(x,y,z) = d(x,z,y) = d(y,x,z) =$$

$$d(y,z,x) = d(z,x,y) = d(z,y,x).$$

(iv) rectangle inequality:

$$d(x,y,z) \leq d(x,y,t) + d(y,z,t) +$$

$$d(z,x,t)$$

for $x,y,z,t \in X$.

Then d is a 2-metric and (X, d) is a 2-metric space.

Definition 1.2. [8] Let X be a non-empty set and $d : X \times X \times X \rightarrow \mathbb{R}_+$ be a map satisfying the following properties

(i) $d(x,y,z) = 0$ if at least two of the three points are the same.

(ii) For $x,y \in X$ such that $x \neq y$ there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.

(iii) symmetry property: for $x,y,z \in X$,

$$d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) =$$

$$d(z,x,y) = d(z,y,x).$$

(iv) s-rectangle inequality: there exists $s \geq 1$ such that

$$d(x,y,z) \leq s[d(x,y,t) + d(y,z,t) + d(z,x,t)]$$

for $x,y,z,t \in X$.

Then d is a b2-metric and (X, d) is a b2-metric space

If $s=1$, the b2-metric reduces to the 2-metric.

Definition 1.3. [10] Let X be a non-empty set and $d : X \times X \times X \rightarrow \mathbb{R}_+$ be a map satisfying the following properties:

(i) $d(x,y,z) = 0$ if at least two of the three points are the same.

(ii) For $x,y \in X$ such that $x \neq y$ there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.

(iii) symmetry property: for $x,y,z \in X$,

$$d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) =$$

$$d(z,x,y) = d(z,y,x).$$

(iv) modified rectangle inequality: there exists $\alpha, \beta, \gamma \geq 1$ such that

$$d(x,y,z) \leq \alpha d(x,y,t) + \beta d(y,z,t) +$$

$$\gamma d(z,x,t)$$

for $x,y,z,t \in X$.

Then d is a generalized b2-metric and (X,d) is a generalized b2- metric space.

If $\alpha = \beta = \gamma = s$ then a generalized b2-metric is a b2-metric. If $\alpha = \beta = \gamma = 1$ then the b2- metric is a 2-metric. The example that follows provides a motivation for the generalization of the concept of a b2-metric.

In recent times, Kamran et al. [19] introduced an expansion of b-metric space known as extended b-metric space.

Definition 1.4. [19] Consider a nonempty set S and a mapping $\varphi : S \times S \rightarrow [1, +\infty)$. A mapping $d_\varphi : S \times S \rightarrow [0, +\infty)$ is known to be an extended b-metric space if it satisfies the succeeding assumptions:

$$(d_\varphi 1) \quad d_\varphi(x, y) = 0 \text{ if } x = y,$$

$$(d_\varphi 2) \quad d_\varphi(x, y) = d_\varphi(x, y),$$

$$(d_\varphi 3) \quad d_\varphi(x, y) \leq \varphi(x, y) \{ d_\varphi(x, z) + d_\varphi(z, y) \}$$

for every $x, y, z \in S$. (S, d_ϕ) is known as an extended b-metric space.

Inspired from Definition 1.3 and Definition 1.4 given above, here in this paper we introduce the concept of generalized extended b2-metric space and we prove a fixed point result in this space.

2. Main Result

In this section, we introduce the following generalized extended b2 metric space and then prove a fixed point result on it.

Definition 2.1 Let X be a non-empty set Let $\rho, \tau, \sigma: X \times X \times X \rightarrow [1, +\infty)$ and $d: X \times X \times X \rightarrow \mathbb{R}_+$ be a map satisfying the following properties:

(i) $d(x,y,z) = 0$ if at least two of the three points are the same.
 (ii) For $x,y \in X$ such that $x \neq y$ there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.

(iii) symmetry property: for $x,y,z \in X$,
 $d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x)$.

(iv) modified rectangle inequality:
 $d(x,y,z) \leq \rho(x,y,z) d(x,y,t) + \tau(x,y,z)d(y,z,t) + \sigma(x,y,z)d(z,x,t)$
 for $x,y,z,t \in X$.

If $\rho(x,y,z) = \alpha, \tau(x,y,z) = \beta, \sigma(x,y,z) = \gamma$ then a generalized extended b2-metric is a generalized b2-metric $\rho(x,y,z) = \tau(x,y,z) = \sigma(x,y,z) = s$ then a generalized extended b2-metric is a b2-metric. If $\rho(x,y,z) = \tau(x,y,z) = \sigma(x,y,z) = 1$ then a generalized extended b2-metric is a 2-metric.

Definition 2.2. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in a generalized b2-metric space (X, d) .

a) the sequence $\{x_n\}_{n \in \mathbb{N}}$ is convergent to $x \in X$ iff for all $z \in X, \lim_{n \rightarrow \infty} d(x_n, x, z) = 0$.

b) the sequence $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X iff for all $z \in X, \lim_{n,m \rightarrow \infty} d(x_n, x_m, z) = 0$.

Definition 2.3: Let $\phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a function satisfying

- i) ϕ is continuous
- ii) $\phi(t) < t$
- iii) $\sum_1^n \phi^i(t) < \infty$
- iv) $\lim_{n \rightarrow \infty} \sum_1^n \phi^i(t) \rightarrow 0$ as $n \rightarrow \infty$.

Theorem 2.5: Let (X,d) be a complete generalized extended b2-metric space and $T: X \rightarrow X$ be a self mapping

(1) $d(Tx, Ty, z) \leq \phi(d(x, y, z))$ (ϕ - contraction)
 for all $x,y,z \in X$. Then T has a fixed point in X . Also assume that $\max \{\rho(x,y,z), \tau(x,y,z), \sigma(x,y,z)\} < 1/k$ where $k \in (0, 1)$.

Proof: Let $x_0 \in X$ and define a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X by $x_n = Tx_{n-1}$, for all $n \in \mathbb{N}$. We shall show that the sequence $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence of real. Using (1), we get

$$d(x_n, x_{n+1}, z) = d(Tx_{n-1}, Tx_n, z) \leq \phi(d(x_{n-1}, x_n, z))$$

which on repeating application implies that

$$d(x_n, x_{n+1}, z) \leq \phi^n(d(x_0, x_1, z))$$

Let $n, m \in \mathbb{N}$ so that $n < m$,

$$\begin{aligned} d(x_n, x_m, z) &\leq \rho(x_n, x_m, z) d(x_n, x_{n+1}, z) + \tau(x_n, x_m, z) d(x_n, x_m, x_{n+1}) + \sigma(x_n, x_m, z) d(x_{n+1}, x_m, z) \\ &\leq \rho(x_n, x_m, z) \phi^n(d(x_0, x_1, z)) + \tau(x_n, x_m, z) \phi^n(d(x_0, x_1, x_m)) \\ &\quad + \sigma(x_n, x_m, z) d(x_{n+1}, x_m, z) \\ &\leq \rho(x_n, x_m, z) \phi^n(d(x_0, x_1, z)) + \tau(x_n, x_m, z) \phi^n(d(x_0, x_1, x_m)) \\ &\quad + \sigma(x_n, x_m, z) (\rho(x_{n+1}, x_m, z) \phi^{n+1}(d(x_0, x_1, z)) + \tau(x_{n+1}, x_m, z) \phi^{n+1}(d(x_0, x_1, x_m)) + \sigma(x_{n+1}, x_m, z) d(x_{n+2}, x_m, z)) \end{aligned}$$

continuing we get

$$\begin{aligned} d(x_n, x_m, z) &\leq \sum_{i=0}^{m-n-1} \sigma(x_{n+i}, x_m, z) (\rho(x_{n+i+1}, x_m, z) \phi^{n+i+1}(d(x_0, x_1, z)) + \tau(x_{n+i+1}, x_m, z) \phi^{n+i+1}(d(x_0, x_1, x_m))) \\ &\quad + \sum_{i=0}^{m-n-1} \sigma(x_{n+i}, x_m, z) (\sigma(x_{n+i+1}, x_m, z) \phi^{n+i+1}(d(x_0, x_1, z))) \\ &\leq \sum_{i=0}^{m-n-1} \frac{1}{k^2} (2\phi^{n+i+1}(d(x_0, x_1, z)) + \phi^{n+i+1}(d(x_0, x_1, x_m))) \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \text{ by condition iv) of Definition 2.3.} \end{aligned}$$

And so $\lim_{n,m \rightarrow \infty} d(x_n, x_m, z) = 0$ which state that $\{x_n\}$ is a Cauchy sequence in complete generalized extended b2-metric space X so it is convergent in X i.e, $\{x_n\}$ converges to some $x \in X$. Now we prove that x is a fixed point of T .

$$\begin{aligned} d(x_n, Tx, z) &= d(Tx_{n-1}, Tx, z) \\ &\leq \phi(d(x_{n-1}, x, z)) \\ &< d(x_{n-1}, x, z) \end{aligned}$$

using condition i) of Definition 2.3

Which on letting $n \rightarrow \infty$ gives $d(x, Tx, z) = 0$ so that $Tx = x$ i.e. x is a fixed point of the mapping T .

For the uniqueness of x , let $x \neq y \in X$ be such that $Ty = y$. Then

$$d(x, y, z) = d(Tx, Ty, z) \leq \phi(d(x, y, z)) < d(x, y, z)$$

which is a contradiction. So $x = y$.

If we take $\phi(t) = kt$ where $k \in (0, 1)$, then ϕ -contraction is a Banach type contraction.

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