DOI: 10.53469/jerp.2025.07(06).10

Comparative Analysis of Fixed-Point Theorems: From Standard Metric to Generalized Extended b²-Metric Spaces with Contractions

Shruti Maheshwari

Department of Mathematics, Government Degree College Nanauta, Saharanpur
shruti@gmail.com

Abstract: ***Z. Mustafa [8] introduced generalized metric space called b2-metric space. Kamran et al. [7], have dealt with an extended b-metric space. The aim of this paper is to establish a notion of a generalized extended b2 metric space which extends and generalizes metric space due to Z. Mustafa [8], Khan et al [10] and Kamran et al. [7]. Also we prove a fixed point theorem on a generalized extended b2 metric space.***

Keywords: metric space, b metric space, 2 metric space, generalized 2 metric space, extended 2 metric space.

2020 AMS Subject Classification: 47H10, 54H25

1. Introduction

The notion of a 2-metric space was introduced by Gahler, in [4]. Several fixed-point results were obtained in [1,2,3,4,5 6], as a generalization of the concept of a metric space. A 2-metric is not a continuous function of its variables, whereas an ordinary metric is. The basic philosophy is that since a 2-metric measures area, a contraction should send the space towards a configuration of zero area, which is to say a line.

Z. Mustafa introduced a new type of generalized metric space called b2-metric space, as a generalization of the 2-metric space, [8].

Recently, Kamran et al., have dealt with an extended b-metric space and obtained unique fixed-point results, [7].

Definition 1.1. [4, 9] Let X be a non-empty set and d :X×X×X → R_+ be a map satisfying the following properties
(i) d(x,y,z) = 0 if at least two of the three points are the same
.
(ii) For x, y ∈ X such that x ≠y there exists a point z ∈X such that d(x,y,z) ≠0.
(iii) symmetry property: for x, y, z ∈ X,
d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x).
(iv) rectangle inequality:
d(x,y,z) ≤ d(x,y,t) + d(y,z,t) + d(z,x,t)
for x,y,z,t ∈ X.
Then d is a 2-metric and (X, d) is a 2-metric space.

Definition 1.2. [8] Let X be a non-empty set and d :X×X×X → R_+ be a map satisfying the
following properties
(i) d(x,y,z) = 0if at least two of the three points are the same.
(ii) For x,y∈X such that x ≠y there exists a point z ∈X such that d(x,y,z) ≠0.
(iii) symmetry property: for x,y,z∈X,
d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x).
(iv) s-rectangle inequality:there exists s ≥1such that
d(x,y,z)≤s[d(x,y,t) + d(y,z,t) + d(z,x,t)]
for x,y,z,t ∈ X.
Then d is a b2-metric and (X, d) is a b2-metric space
If s=1, the b2-metric reduces to the 2-metric.

Definition 1.3. [10] Let X be a non-empty set and d :X×X×X → R_+ be a map satisfying the following properties:
(i) d(x,y,z) = 0 if at least two of the three points are the same.
(ii) For x,y∈X such that x ≠y there exists a point z ∈X such that d(x,y,z) ≠0.
(iii) symmetry property: for x,y,z∈X,
d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x).
(iv) modified rectangle inequality: there exists α, β, γ≥1such that
d(x,y,z) ≤ αd(x,y,t) + βd(y,z,t) + γd(z,x,t)]
for x,y,z,t ∈ X.

Then d is a generalized b2-metric and (X,d)is a generalized b2- metric space.

If α = β = γ = s then a generalized b2-metric is a b2-metric. If α = β = γ = 1 then the b2- metric is a 2-metric. The example that follows provides a motivation for the generalization of the concept of a b2-metric.

In recent times, Kamran et al. [19] introduced an expansion of b-metric space known as extended b-metric space.

Definition 1.4. [19] Consider a nonempty set S and a mapping φ : S × S → [1, +∞). A mapping $d_φ$: S × S → [0, +∞) is known to be an extended b-metric space if it satisfies the succeeding assumptions:
($d_φ$1) $d_φ$(x, y) =0 if x = y,
($d_φ$ 2) $d_φ$(x, y) = $d_φ$(x, y),
($d_φ$ 3) $d_φ$(x, y) ≤ φ(x, y) { $d_φ$(x, z) + $d_φ$(z, y)}

for every x, y, z ∈ S. (S, d_φ) is known as an extended b-metric space.

Inspired from Definition 1.3 and Definition 1.4 given above, here in this paper we introduce the concept of generalized extended b2-metric space and we prove a fixed point result in this space.

2. Main Result

In this section, we introduce the following generalized extended b2 metric space and then prove a fixed point result on it.

Definition 2.1 Let X be a non-empty set Let ρ, τ, σ: X×X×X → [1, +∞) and d :X×X×X → R_+ be a map satisfying the following properties:
(i) d(x,y,z) = 0 if at least two of the three points are the same.
(ii) For x,y∈X such that x ≠y there exists a point z ∈X such that d(x,y,z) ≠0.
(iii) symmetry property: for x,y,z ∈ X,
d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x).
(iv) modified rectangle inequality:
d(x,y,z) ≤ ρ(x,y,z) d(x,y,t) + τ(x,y,z)d(y,z,t) + σ(x,y,z)d(z,x,t)]
for x,y,z,t ∈ X.

If ρ(x,y,z) = α, τ(x,y,z) = β, σ(x,y,z) = γ then a generalized extended b2-metric is a generalized b2-metric ρ(x,y,z) = τ(x,y,z) = σ(x,y,z) =s then a generalized extended b2-metric is a b2-metric. If ρ(x,y,z) = τ(x,y,z) = σ(x,y,z) = 1 then a generalized extended b2-metric is a 2-metric.

Definition 2.2. Let $\{x_n\}$n ∈ N be a sequence in a generalized b2-metric space (X, d).
a) the sequence $\{x_n\}$n ∈ N is convergent to x ∈X iff for all z ∈ X , $\lim_{n\to\infty} d(x_n,x,z) = 0$.
b) the sequence $\{x_n\}$n ∈ N is a Cauchy sequence in X iff for all z ∈ X, $\lim_{n,m\to\infty} d(x_n,x_m,z) = 0$.

Definition 2.3: Let φ : $R_+ \to R_+$ be a function satisfying
i) φ is continuous
ii) $\varphi(t) < t$
iii) $\sum_1^n \varphi^i (t) < \infty$
iv) $\lim_{n\to\infty} \sum_1^n \varphi^i (t) \to 0$ as $n \to \infty$.

Theorem 2.5: Let (X,d)be a complete generalized extended b2-metric space and T :X→X be a self mapping
(1) d(Tx, Ty, z) ≤ φ(d(x, y, z)) (φ- contraction)
for all x,y,z ∈ X. Then T has a fixed point in X. Also assume that max {ρ(x,y,z), τ(x,y,z) σ(x,y,z)} < 1/k where k ∈ (0, 1).

Proof: Let x_0 ∈ X and define a sequence $\{x_n\}$ n ∈ N in X by $x_n=Tx_{n-1}$, for all n ∈ N. We shall show that the sequence $\{x_n\}$ n ∈ N is a Cauchy sequence of real. Using (1), we get
$d(x_n, x_{n+1}, z) = d(Tx_{n-1}, Tx_n, z) \leq \varphi(d(x_{n-1}, x_n, z))$
which on repeating application implies that

$d(x_n, x_{n+1}, z) \leq \varphi^n(d(x_0, x_1, z))$
Let n, m ∈ N so that n < m,

$$d(x_n, x_m, z) \leq \rho(x_n, x_m, z)\, d(x_n, x_{n+1}, z) + \tau(x_n, x_m, z)d(x_n, x_m, x_{n+1}) + \sigma(x_n, x_m, z)d(x_{n+1}, x_m, z)$$
$$\leq \rho(x_n, x_m, z)\, \varphi^n(d(x_0, x_1, z)) + \tau(x_n, x_m, z)\, \varphi^n(d(x_0, x_1, x_m)) + \sigma(x_n, x_m, z)d(x_{n+1}, x_m, z)$$
$$\leq \rho(x_n, x_m, z)\, \varphi^n(d(x_0, x_1, z)) + \tau(x_n, x_m, z)\, \varphi^n(d(x_0, x_1, x_m)) + \sigma(x_n, x_m, z)\,(\,\rho(x_{n+1}, x_m, z)\, \varphi^{n+1}(d(x_0, x_1, z)) + \tau(x_{n+1}, x_m, z)\, \varphi^{n+1}(d(x_0, x_1, x_m)) + \sigma(x_{n+1}, x_m, z)d(x_{n+2}, x_m, z))$$

continuing we get

$$d(x_n, x_m, z) \leq \sum_{i=0}^{m-n-1} \sigma(x_{n+i}, x_m, z)\,(\,\rho(x_{n+i+1}, x_m, z)\, \varphi^{n+i+1}(d(x_0, x_1, z)) + \sum_{i=0}^{m-n-1} \sigma(x_{n+i}, x_m, z)\,(\tau(x_{n+i+1}, x_m, z)\, \varphi^{n+i+1}(d(x_0, x_1, x_m)) + \sum_{i=0}^{m-n-1} \sigma(x_{n+i}, x_m, z)\,(\,\sigma(x_{n+i+1}, x_m, z)\, \varphi^{n+i+1}(d(x_0, x_1, z))$$
$$\leq \sum_{i=0}^{m-n-1} \frac{1}{k^2}\ \left(2\varphi^{n+i+1}\big(d(x_0, x_1, z)\big) + \varphi^{n+i+1}(d(x_0, x_1, x_m)\right)$$
$\to 0$ as $n \to \infty$ by condition iv) of Definition 2.3.

And so $\lim_{n,m\to\infty} d(x_n, x_m, z) = 0$ which state that $\{x_n\}$ is a Cauchy sequence in complete generalized extended b2-metric space X so it is convergent in X i,e, $\{x_n\}$ converges to some x ∈ X. Now we prove that x is a fixed point of T.
$d(x_n, Tx, z) = d(Tx_{n-1}, Tx, z)$
$\leq \varphi(d(x_{n-1}, x, z))$
$< d(x_{n-1}, x, z)$
using condition i) of Definition 2.3

Which on letting $n \to \infty$ gives d(x, Tx, z) =0 so that Tx = x i.e. x is a fixed point of the mapping T.

For the uniqueness of x, let x ≠ y ∈ X be such that Ty = y. Then
d(x, y, z) = d(Tx, Ty, z) ≤ φ(d(x, y, z)) < d(x, y, z)

which is a contradiction. So x = y.

If we take φ(t) = kt where k ∈ (0, 1), then φ -contraction is a Banach type contraction.

References

[1] A. Aghajani, M. Abbas, J. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, Math. Slovaca, 64 (2014), 941–960. https://doi.org/10.2478/s12175-014-0250-6.
[2] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostrav. 1 (1993), 5–11. http://dml.cz/dmlcz/120469.
[3] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, Atti Sem. Mat. Fis. Univ. Modena, 46 (1998), 263–276. https://cir.nii.ac.jp/crid/1571980075066433280.
[4] S. Gahler, 2-metrische Raume und ihre topologische Struktur, Math. Nachr. 26 (1963), 115–148. https: //doi.org/10.1002/mana.19630260109.
[5] M. Geraghty, On contractive mappings, Proc. Amer. Math. Soc. 40 (1993), 604–608.

[6] T.L. Hicks, B.E. Rhoades, A Banach type fixed point theorem, Math. Japon. 24 (1979), 327–330. https://cir.nii.ac.jp/crid/1572824499579575040.

[7] T. Kamran, M. Samreen, Q. UL Ain, A generalization of b-metric space and some fixed point theorems, Mathematics, 5 (2017), 19. https://doi.org/10.3390/math5020019.

[8] Z. Mustafa, V. Parvaneh, J.R. Roshan, et al. b2-Metric spaces and some fixed point theorems, Fixed Point Theory Appl. 2014 (2014), 144. https://doi.org/10.1186/1687-1812-2014-144.

[9] P. Singh, V. Singh, S. Singh, Some fixed points results using (ψ,φ)-generalized weakly contractive map in a generalized 2-metric space, Adv. Fixed Point Theory, 13 (2023), 21. https://doi.org/10.28919/afpt/8218.

[10] S.H. Khan, P. Singh, S. Singh, et al. Fixed point results in generalized bi-2-metric spaces using θ-type contractions, Contemp. Math. 5 (2024), 1257–1272

[11] O. Ege and I. Karaca, Common fixed point results on complex valued Gb-metric spaces, Thai J. Math. 16, 3, 775-787, 2018.

[12] O. Ege, C. Park and A. H. Ansari, A different approach to complex valued Gb-metric spaces, Adv. Difference Equ. 2020, 152, 1-13, 2020.

[13] A. Gholidahneh, S. Sedghi, O. Ege, Z. D. Mitrovic and M. de la Sen, The Meir-Keeler type contractions in extended modular b-metric spaces with an application, AIMS Math. 6, 2, 1781-1799, 2021.

[14] V. Gupta, O. Ege, R. Saini and M. de la Sen, Various fixed point results in complete Gb-metric spaces, Dynam. Systems Appl. 30, 2, 277-293, 2021.

[15] A. Hassen, R. Dusan, A. Aghajani, T. Dosenovic, M. Salmi, M. Nooraniand and H. Qawaqneh, On fixed point results in Gb-metric spaces, Mathematics, 2019, 7, 617, doi:10.3390/math7070617.

[16] M. Iqbal, A. Batool, O. Ege and M. de la Sen, Fixed point of almost contraction in b-metric spaces, J. Math. 3218134, 1-6, 2020.

[17] M. Iqbal, A. Batool, O. Ege and M. de la Sen, Fixed point of generalized weak contraction in b-metric spaces, J. Funct. Spaces 2021, 2042162, 1-8, 2021.

[18] T. Kamran, M. Samreen and O.U Ain, A generalization of b-metric space and some fixed point theorems, Mathematics 5, 2, 2017.