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Exploring Properties and Classifications of Heron Triangles in Mathematics

Sonam Nidhi

Mindanao State University-Buug Campus, Laboratory High School, Datu Panas, Buug, Zamboanga Sibugay, Philippines
sonam2019@gmail.com

Abstract: ***This study explores Heron triangles, which are unique triangles characterized by integer side lengths and areas. It classifies various types, including Pythagorean, consecutive, isosceles, and integral Heron triangles, and examines their mathematical properties such as semiperimeters, inradii, circumscribed radii, and heights. By presenting definitions, illustrations, and detailed proofs of theorems, this paper aims to deepen understanding and provide a foundation for further mathematical applications.***

Keywords: Heron triangle, integer geometry, Pythagorean triangle, mathematical proofs, triangle properties

1. Introduction

1.1 Background of the Study

A Heron triangle is triangle such that the lengths of its three sides as well as its area are integers. The name refers to the formula for the area of a triangle given the sides known as Heron's formula. This remarkable formula is the square root of the product of the four factors, one of which is the semiperimeter of the triangle (that is, one-half the sum of the three sides), and the other three factors are obtained by subtracting each side from the semiperimeter. The simplest example of a Heron triangle is a right triangle.

This research builds upon the study entitled 'On Heron Triangle' by J´osef S´andor of the Department of Mathematics, Babes-Bolyai University, Cludy Napoka, Romania

1.2 Statement of the Problem

In this paper, some properties of a triangle that satisfies the required conditions of a Heron triangle as well as its characteristics are identified.

1.3 Objectives of the Study

This paper aims to classify and analyze the properties of Heron triangles, contributing to the understanding of their unique characteristics and potential application in mathematical studies.

1.4 Significance of the Study

This study contributes to the field of geometry by providing a detailed exploration of Heron triangles, offering insights into their classifications and properties, and creating a basis for future research in both theoretical and applied mathematics.

1.5 Methodology

This paper is expository in nature. Definitions and concepts are being presented in order to support some results. Details of the proof of some results are supplied for clarification. Examples are also provided to verify the results for simple cases.

2. Preliminary Notion and Results

2.1 Basic Concepts

Definition 2.1.1 Pythagorean triangle is a right triangle whose sides are of integral length. It is denoted as P-triangle.

Theorem 2.1.2 (Pythagorean Theorem and its Converse) [4] In a right triangle, c is the length of the hypotenuse, a and b are the lengths of the legs if and only if $c^2 = a^2 + b^2$.

Definition 2.1.3 Pythagorean numbers are numbers satisfying the equation $c^2 = a^2 + b^2$ where $a, b,$ and c are the sides of the P-triangle. The set $\{a, b, c\}$ is the set of Pythagorean numbers.

Example 2.1.4 The sets $\{3, 4, 5\}$, {5, 12, 13}, {6, 8, 10}, {7, 24, 25}, {8, 15, 17}, {9, 12, 15} are sets of Pythagorean numbers. In fact,

$$(1)\ \left\{n, \frac{n^2-4}{4}, \frac{n^2+4}{4}\right\}$$

when n is even is a set of Pythagorean numbers. Also,

$$(2)\ \left\{n, \frac{n^2-1}{2}, \frac{n^2+1}{2}\right\}$$

when n is odd is also a set of Pythagorean numbers.

Example 2.1.5 Consider $n = 12$ in formula (1). Then the numbers 35 and 37 are the values of $\frac{n^2-4}{4}$ and $\frac{n^2+4}{4}$, respectively. That is, for $n = 12$,

$$\frac{n^2-4}{4} = \frac{12^2-4}{4} = \frac{144-4}{4} = \frac{140}{4} = 35 \text{ and}$$
$$\frac{n^2+4}{4} = \frac{12^2+4}{4} = \frac{144+4}{4} = \frac{148}{4} = 37.$$

Now,

$$122 + 352 = 372$$
$$144 + 1225 = 1369$$
$$1369 = 1369$$

Thus, the set {12, 35, 37} is a set of Pythagorean numbers.

Example 2.1.6 Consider $n = 9$ in formula (2). Then the numbers 40 and 41 are the values of $\frac{n^2-1}{2}$ and $\frac{n^2+1}{2}$, respectively. That is, for $n = 9$,

$$\frac{n^2-1}{2} = \frac{9^2-1}{2} = \frac{81-1}{2} = \frac{80}{2} = 40 \text{ and}$$
$$\frac{n^2+1}{2} = \frac{9^2+1}{2} = \frac{81+1}{2} = \frac{82}{2} = 41.$$

Now,

$$92 + 402 = 412$$
$$81 + 1600 = 1681$$
$$1681 = 1681.$$

Hence, the set {9, 40, 41} is a set of Pythagorean numbers.

Definition 2.1.7 The semiperimeter p of a triangle is one-half of the triangle's perimeter. That is, for a triangle with sides of lengths a, b, and c,

$$p = \tfrac{1}{2}(a + b + c).$$

Definition 2.1.8 An integer b is said to be divisible by an integer $a \neq 0$, in symbols $a|b$, if there exists some integer c such that $b = ac$. We write $a \nmid b$ to indicate that b is not divisible by a.

Definition 2.1.9 Let a and b be given integers, with at least one of them different from zero. The greatest common divisor of a and b is the positive integer d denoted by $(a, b) = d$, if the following are satisfied:
(a) $d|a$ and $d|b$.
(b) If $c|a$ and $c|b$, then $c \leq d$.

Definition 2.1.10 Let $a, b \in Z$, then a and b are relatively prime if in case $(a, b) = 1$. Furthermore $a_1, a_2, \dots, a_n \in Z$ are relatively prime in case $(a_1, a_2, \dots, a_n) = 1$. Thus, $a_1, a_2, \dots, a_n$ are relatively prime in pairs in case $(a_i, a_j) = 1$ for all $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$ with $i \neq j$. If $(a, b) = 1$, then a and b are coprime, or a is coprime of b.

Definition 2.1.11 If two integers are both odd or both even, they are said to have the same parity. If one is odd and the other is even, then they have different parities.

Definition 2.1.12 An angle bisector is a line segment that cuts the angle into two congruent angles.

Definition 2.1.13 A perpendicular bisector is a line that forms a right angle with one of the triangle's sides and intersects that side at its midpoint.

Definition 2.1.14 The inscribed circle or the incircle is the largest possible circle that can be drawn interior to a triangle of which each side of the triangle is tangent to the circle. The center of the inscribed circle can be found as the intersection of the three internal angle bisectors which is called the incenter. The radius of the inscribed circle is the inradius of the triangle. The inradius r is given by $r = p - c$ where p is the semiperimeter and c is the hypotenuse of the triangle or $r = A/p$ where A is the area and p is the semiperimeter of the triangle.

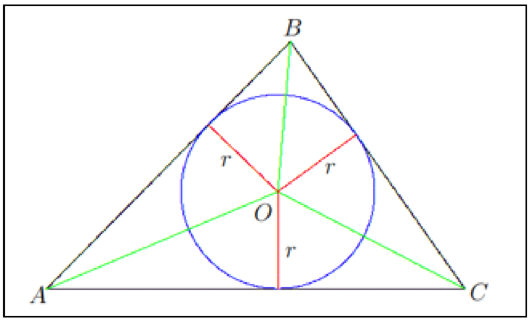

Figure 2.1: Inscribed circle of a triangle, the angle bisector (green), the in radius (r), and the incenter (O)

Definition 2.1.15 The circumscribed circle or the circumcircle of the triangle is a circle that passes through all the vertices of the triangle. The center of this circle is called the circumcenter. The circumcenter of a triangle can be found as the intersection of the three perpendicular bisectors. The radius of the circumscribed circle is denoted by R and given by $R = \frac{1}{2}c$ where c is the diameter of the circumscribed circle and is the hypotenuse of the P-triangle or $R = abc/A$ where $a, b,$ and c are the sides and A is the area of the triangle.

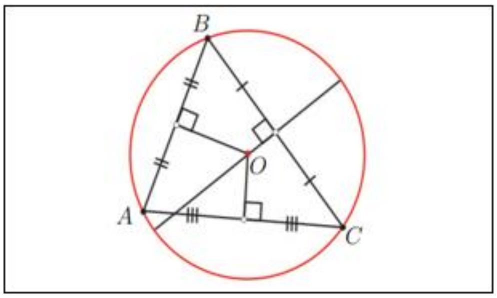

Figure 2.2: Circumference of a triangle (red) and the circumcenter (O)

Definition 2.1.16 An exscribed circle of a triangle is a circle lying outside the triangle, tangent to one of its sides and tangent to the extensions of the other two sides. Every triangle has three sides' distinct exscribed circles each tangent of one of the triangle's sides. The center of the exscribed circle is the intersection of the internal bisector of one angle and the internal bisector of the other two. The radius of the exscribed circle with respect to side b is denoted by r_b which is given by $r_b = A/(p - b)$ where A is area and p is semiperimeter of the triangle.

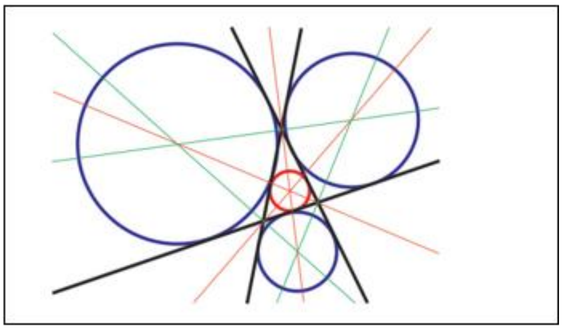

Figure 2.3: Exscribed circle of a triangle (blue circles), Inscribed circle (red lines) and external angle bisectors (green lines)

Definition 2.1.17 The equation $x^2 - dy^2 = N$, with given integers d and N and unknowns x and y, is usually called Pell's equation. If d is negative, it can have a finite number of solutions. If d is a perfect square, say $d = a^2$, the equation reduces to $(x - ay)(x + ay) = N$ and again there is only a finite number of solutions.

Theorem 2.1.18 [1] All the solutions of the Pythagorean equation $x^2 + y^2 = z^2$, satisfying the conditions $gcd(x, y, z) = 1, 2|x, x > 0, y > 0, z > 0$, are given by the formulas $x = 2st$, $y = s^2 - t^2$, $z = s^2 + t^2$, for integers $s > t > 0$ such that $gcd(s, t) = 1$ and s is not congruent to $t(mod\ 2)$.

Theorem 2.1.19 [1] Let x_1, y_1 be the fundamental solution of $x^2 - dy^2 = N$. Then every pair of integers x_n, y_n defined by the condition $x_n + y_n\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^n$, $n = 1, 2, 3, \ldots$ is also a positive solution.

Remark 2.1.20 From $x^2 - dy^2 = 1$ of Theorem 2.1.19, note that since $(x_1, y_1) = (2, 1)$ then $x_n + y_n\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^n$, $n = 1, 2, 3, \ldots$.

2.2 Review of Related Literature

Remark 2.2.1 The sum of two numbers of different parities is odd.

Proof : Let x and y be the two numbers. Suppose x is even and y is odd, then $x = 2a$ and $y = 2b + 1$ for some $a, b, \in Z$. Thus, $x + y = 2a + 2b + 1 = 2(a + b) + 1 = 2c + 1$ where $c = a + b \in Z$. Thus, $x + y = 2c + 1$ which is odd.

Remark 2.2.2 The square of an odd integer is odd and the square of an even integer is even.

Proof : Let x be an odd integer. Then $x = 2a + 1$ for some $a \in Z$. Thus, $x^2 = (2a + 1)^2 = 4a^2 + 4a + 1 = 2(2a^2 + 2a) + 1 = 2b + 1$ where $b = 2a^2 + 2a \in Z$. Thus, $x^2 = 2b + 1$ which is odd. Similarly, let $y = 2c$ for some $c \in Z$. Then $y^2 = (2c)^2 = 4c^2 = 2(4c^2) = 2d$ where $d = 2c^2 \in Z$. Hence, $y^2 = 2d$ which is even.

Remark 2.2.3 Let a, b be integers where one is odd and the other is even, then $(a, b) = 1$.

Proof : Without loss of generality, let x be the even integer and y be the odd integer. Suppose $(x, y) = d$ then $d|x$ and $d|y$. Since x is even then d must also be even. Also, since y is odd then d must also be odd. This implies that d must be equal to 1.

3. Results and Discussion

3.1 Basic Concepts of Heron Triangle

This section deals with the basic concepts needed to identify the properties of a triangle that satisfies the required condition of a Heron triangle.

Definition 3.1.1 A Heron triangle, denoted by H-triangle, is a triangle such that the lengths of its three sides as well as its area are integers.

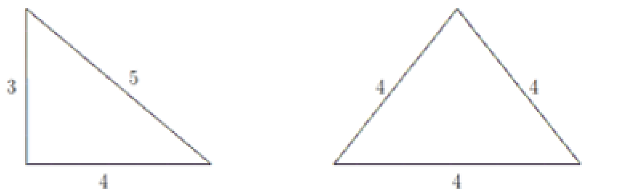

Figure 3.1.a
A Heron triangle with $A = \frac{1}{2}(3)(4) = 6$

Figure 3.1.b
A triangle which is not a Heron triangle with $A = \frac{1}{2}(2\sqrt{3})(4) = 4\sqrt{3}$

Theorem 3.1.2 If a triangle has sides of lengths a, b, c and if p = $\frac{1}{2}$(a+b+c), the "semiperimeter" of the triangle, then Heron's formula says that the area A of the triangle is given by

$$A = \sqrt{p(p-a)(p-b)(p-c)}.$$

Illustration 3.1.3

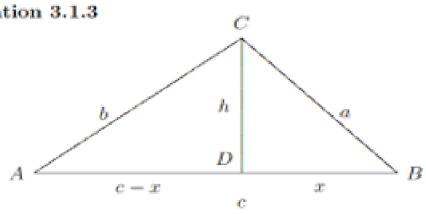

Figure 3.2: A ΔABC with $\overline{CD}$ perpendicular to $\overline{AB}$

Proof : Let $a, b,$ and c be the sides of ΔABC and let $\overline{CD}$ be perpendicular to $\overline{AB}$ where $|CD| = h$ and $|DB| = x$ (see Figure 3.2). Then $a^2 = x^2 + h^2$ and $b^2 = (c - x)^2 + h^2$ by the Pythagorean Theorem. Solving for h^2 in both equations, give

$$h^2 = a^2 - x^2, \quad (3.1)$$

from the first equation, and

$$h^2 = b^2 - (c - x)^2$$
$$h^2 = b^2 - (c^2 - 2cx + x^2)$$
$$h^2 = b^2 - c^2 + 2cx - x^2, \quad (3.2)$$

from the second equation.

Subtracting equation (3.1) from (3.2), yields to

$$h^2 - h^2 = b^2 - c^2 + 2cx - x^2 - (a^2 - x^2)$$
$$0 = b^2 - c^2 + 2cx - x^2 - a^2 + x^2$$
$$0 = b^2 - c^2 + 2cx - a^2.$$

Solving for x gives

$$2cx = -b^2 + c^2 + a^2$$
$$\frac{2cx}{2c} = \frac{-b^2 + c^2 + a^2}{2c}$$
$$x = \frac{-b^2 + c^2 + a^2}{2c}.$$

Now, the area of a triangle is A = $\frac{1}{2}$(base)(height), so A = $\frac{1}{2}ch$.

Solving for h,

$$A = \frac{1}{2}ch$$
$$2A = ch$$
$$h = \frac{2A}{c}.$$

From $h^2 = a^2 - x^2$, substitute $x = \frac{-b^2+c^2+a^2}{2c}$ and $h = \frac{2A}{c}$, to obtain

$$\left(\frac{2a}{c}\right)^2 = a^2 - \left(\frac{a^2-b^2+c^2}{2c}\right)^2$$
$$\frac{4A^2}{c^2} = a^2 - \frac{\left(a^2-b^2+c^2\right)^2}{4c^2}$$
$$\frac{4A^2}{c^2} = \frac{4a^2c^2-\left(a^2-b^2+c^2\right)^2}{4c^2}.$$

Multiplying by $4c^2$,

$$\left(\frac{4A^2}{c^2}\right)\cdot 4c^2 = \left(\frac{4a^2c^2-(a^2-b^2+c^2)^2}{4c^2}\right)\cdot 4c^2$$
$$16A^2 = 4a^2c^2 - (a^2-b^2+c^2)^2$$
$$16A^2 = (2ac)^2 - (a^2-b^2+c^2)^2$$
$$16A^2 = (2ac - a^2 + b^2 - c^2)(2ac + a^2 - b^2 + c^2)$$
$$16A^2 = [b^2 - (a^2 - 2ac + c^2)][(a^2 + 2ac + c^2) - b^2]$$
$$16A^2 = [b^2 - (a-c)^2][(a+c)^2 - b^2]$$
$$16A^2 = (b-a+c)(b+a-c)(a+c-b)(a+c+b)$$

Dividing by 16 gives

$$\frac{16A^2}{16} = \frac{(b-a+c)(b+a-c)(a+c-b)(a+c+b)}{16}$$
$$A^2 = \left(\frac{(b-a+c)}{2}\right)\left(\frac{(b+a-c)}{2}\right)\left(\frac{(a+c-b)}{2}\right)\left(\frac{(a+c+b)}{2}\right)$$
$$A^2 = \left(\frac{(b-a+c)}{2} + a - a\right)\left(\frac{(b+a-c)}{2} + c - c\right)\left(\frac{(a+c-b)}{2} + b - b\right)\left(\frac{(a+c+b)}{2}\right)$$
$$A^2 = \left(\frac{b-a+c+2a-2a}{2}\right)\left(\frac{b+a-c+2c-2c}{2}\right)\left(\frac{a+c-b+2b-2b}{2}\right)\left(\frac{(a+c+b)}{2}\right)$$
$$A^2 = \left(\frac{a+b+c}{2} - a\right)\left(\frac{a+b+c}{2} - c\right)\left(\frac{a+b+c}{2} - b\right)\left(\frac{(a+c+b)}{2}\right)$$
$$A^2 = (p-a)(p-c)(p-b)p, \text{ where } p = \frac{a+b+c}{2}.$$

So

$$A = \sqrt{p(p-a)(p-b)(p-c)}.$$

Lemma 3.1.4 The general solution of the equation $a^2 + b^2 = c^2$ are given by $a = \lambda(m^2 - n^2)$, $b = 2\lambda mn$ and $c = \lambda(m^2 + n^2)$ (if b is even) where λ is an arbitrary positive integer, while $m > n$ are relatively prime of different parities (that is, $(m, n) = 1$ and m and n cannot be both odd or even).

Proof : It is just sufficient to check that if $a = \lambda(m^2 - n^2)$, $b = 2\lambda mn$, and $c = \lambda(m^2 + n^2)$ then $a^2 + b^2 = c^2$ is satisfied. Thus,

$$\begin{aligned} a^2 + b^2 &= \lambda^2(m^2 - n^2)^2 + (2\lambda mn)^2 \\ &= \lambda^2(m^4 - 2m^2n^2 + n^2) + 4\lambda^2m^2n^2 \\ &= \lambda^2m^4 - 2\lambda^2m^2n^2 + \lambda^2n^2 + 4\lambda^2m^2n^2 \\ &= \lambda^2(m^4 + 2m^2n^2 + n^2) \\ &= \lambda^2(m^2 + n^2)^2 \\ &= [\lambda(m^2 + n^2)]^2 \\ &= c^2 \end{aligned}$$

Hence, the numbers $a = \lambda(m^2 - n^2)$, $b = 2\lambda mn$, and $c = \lambda(m^2 + n^2)$ which are the general solution of $a^2 + b^2 = c^2$ are considered as Pythagorean numbers. □

3.2 Charaterization of a Heron Triangle

This section deals with the types of triangles classified as Heron. These are as follows: the simplest example of a Heron triangle, the P-triangle; the consecutive Heron triangle; an isosceles triangle; and the integral triangle.

3.2.1 The Pythagorean Triangle

Theorem 3.2.1 Let $a = \lambda(m^2 - n^2)$, $b = 2\lambda mn$, and $c = \lambda(m^2 + n^2)$ be the length of the sides of a P-triangle ABC where $\overline{AB}$ is the hypotenuse and λ is an arbitrary positive integer, while $m > n$ are relatively prime of different parities (that is, $(m, n) = 1$ and m and n cannot be both odd or even). Then ΔABC is a Heron triangle.

Illustration:

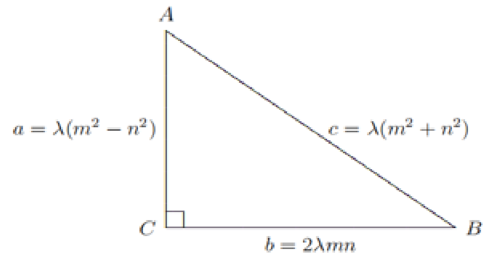

Figure 3.3: A P-triangle ABC

Proof: Clearly $a, b,$ and c are integers since they are sides of a P-triangle. To show that A is an integer:

$$A = \frac{bh}{2} = \frac{ab}{2}, \text{ where } a = h$$
$$A = \frac{\lambda(m^2 - n^2)(2\lambda mn)}{2} = \frac{2\lambda^2 mn(m^2 - n^2)}{2}$$
$$A = \lambda^2 mn(m^2 - n^2)$$

The result follows. □

Consider other characteristics of a Heron triangle. From Theorem 3.1.2, the following corollaries will follow:

Corollary 3.2.2 Let p be the semiperimeter and r be the inradius of a Heron triangle where $p = \frac{a+b+c}{2}$, $r = p - c$, then p and r are integers.

Proof : Let p be the semiperimeter of the ΔABC in Figure 3.3. Then

$$p = \frac{a + b + c}{2}$$
$$p = \frac{\lambda(m^2 - n^2) + 2\lambda mn + \lambda(m^2 + n^2)}{2}$$
$$p = \frac{\lambda m^2 - \lambda n^2 + 2\lambda mn + \lambda m^2 + \lambda n^2}{2}$$
$$p = \frac{2\lambda m^2 + 2\lambda mn}{2} = \frac{2\lambda(m^2 + mn)}{2} = \lambda(m^2 + mn),$$

Implying that p is an integer.

Let r be the inradius of ΔABC. Then

$$r = p - c$$
$$r = \lambda(m^2 + mn) - \lambda(m^2 + n^2)$$
$$r = \lambda m^2 + \lambda mn - \lambda m^2 - \lambda n^2$$
$$r = \lambda mn - \lambda n^2 = \lambda(mn - n^2),$$

implying that r is always an integer.

Since the area A of ΔABC in Figure 3.3 is an integer then it is a Heron triangle. Consider another consequence of Theorem 3.2.1.

Corollary 3.2.3 Let R be the radius of the circumscribed circle and h_a, h_b, h_c be the heights of a Heron ΔABC in

Figure 3.3, where $R = \frac{1}{2}c$ and $h_a = \frac{2A}{a},\ h_b = \frac{2B}{a}, h_c = \frac{2C}{a}$. Then R is an integer if λ is even and the heights h_a, h_b, h_c are all integers if $c|ab$.

Proof : Let R be the radius of the circumscribed circle. Then
$R = \frac{1}{2}c$ (where c is the diameter of the circumscribed circle and is the hypotenuse of the P-triangle)
$R = \frac{\lambda(m^2+n^2)}{2}$ which is an integer when λ is even.

Now, since $h_a = \frac{2A}{a},\ h_b = \frac{2B}{a}, h_c = \frac{2C}{a}$ are the heights of a P-triangle, then
h_a (height with respect to side a)
$= \frac{2A}{a} = \left(\frac{2}{a}\right)\left(\frac{ab}{2}\right) = b = 2\lambda mn,$
h_b (height with respect to side b)
$= \frac{2A}{b} = \left(\frac{2}{b}\right)\left(\frac{ab}{2}\right) = a = \lambda(m^2 - n^2),$
h_c (height with respect to side c)
$= \frac{2A}{a} = \left(\frac{2}{c}\right)\left(\frac{ab}{2}\right) = \frac{ab}{c} = \frac{2\lambda mn(m^2-n^2)}{m^2+n^2}.$
Therefore all heights are integers only if $c|ab$. □

Theorem 3.2.4 In a P-triangle with sides a, b, and c, the quantities A, h_a, h_b, h_c, r and R are integers at the same time if and only if a, b, c are given by $a = 2d(m^4 - n^4)$, $b = 4dmn(m^2 + n^2)$, $c = 2d(m^2 + n^2)^2$ where λ is even, $(m, n) = 1$, and m and n with $m > n$ are of different parities.

Proof: (⇐)Assume that $a = 2d(m^4 - n^4)$, $b = 4dmn(m^2 + n^2)$, and $c = 2d(m^2 + n^2)^2$. Then

$$A = \frac{ab}{2} = \frac{2d(m^4 - n^4) \cdot 4dmn(m^2 + n^2)}{2}$$
$$= 4d^2mn(m^4 - n^4)(m^2 + n^2)$$
$$h_a = \frac{2A}{a} = \frac{8dmn(m^4-n^4)(m^2+n^2)}{2d(m^4-n^4)} = 4mn(m^2 + n^2)$$
$$h_b = \frac{2A}{b} = \frac{8dmn(m^4-n^4)(m^2+n^2)}{4dmn(m^2+n^2)} = 2(m^4 - n^4)$$
$$h_c = \frac{2A}{c} = \frac{8dmn(m^4-n^4)(m^2+n^2)}{2d(m^2+n^2)^2}$$
$$= \frac{8dmn(m^2+n^2)(m^2-n^2)(m^2+n^2)}{2d(m^2+n^2)^2} = 4mn(m^2 - n^2)$$
$$r = p - c$$
$$= \frac{2d(m^4-n^4)+4dmn(m^2+n^2)+2d(m^2+n^2)^2}{2} - 2d(m^2 + n^2)^2$$
$$= 2d(m^4 - n^4) + 2dmn(m^2 + n^2) + d(m^2 + n^2)^2$$
$$-2d(m^2 + n^2)^2$$
$$= 2d(m^4 - n^4) + 2dmn(m^2 + n^2) - d(m^2 + n^2)^2$$
$$R = \frac{c}{2} = \frac{2d(m^2+n^2)^2}{2} = d(m^2 + n^2)^2.$$
Thus, A, h_a, h_b, h_c, r, R are integers at the same time.

(⇒) Assume that A, h_a, h_b, h_c , r , R are all integers. It suffices to show only that $a^2 + b^2 = c^2$ is satisfied by $a = 2d(m^4 - n^4), b = 4dmn(m^2 + n^2), c = 2d(m^2 + n^2)^2$.

$$a^2 + b^2 = 4d^2(m^4 - n^4)^2 + 16d^2m^2n^2(m^2 + n^2)^2$$
$$= 4d^2(m^8 - 2m^4n^4 + n^8)$$
$$+16d^2m^2n^2(m^4 + 2m^2n^2 + n^4)$$
$$= 4d^2m^8 - 8d^2m^4n^4 + 4d^2n^8$$
$$+16d^2m^6n^2 + 32d^2m^4n^4 + 16d^2m^2n^6$$
$$= 4d^2(m^8 + 6m^4n^4 + n^8 + 4m^6n^2 + 4m^2n^6)$$
$$= 4d^2(m^8 + 4m^6n^2 + 6m^4n^4 + 4m^2n^6 + n^8)$$
$$= 4d^2(m^4 + 2m^2n^2+n^4)^2 = 4d^2[(m^2 + n^2)^2]^2$$
$$= [2d(m^2 + n^2)^2]^2 = c^2.$$
Therefore, the result follows. □

3.2.2 Consecutive Heron Triangle

Definition 3.2.5 A consecutive Heron triangle, denoted by CH-triangle, is a Heron triangle whose sides are consecutive integers.

Theorem 3.2.6 Let $2y - 1, 2y, 2y + 1$ be the sides of a triangle ABC where y is a positive integer. Then ΔABC is a Heron triangle specifically known as CH-triangle.

Illustration:

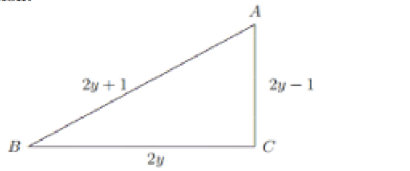

Figure 3.4: A ΔABC with consecutive sides $2y - 1$, $2y$ and $2y + 1$.

Proof : To show that A is an integer in order for ΔABC to be H-triangle. Let $2y - 1, 2y\ and\ 2y + 1$ be the sides of ΔABC. Then
$$p = \frac{(2y-1) + (2y) + (2y+1)}{2} = \frac{2y - 1 + 2y + 2y + 1}{2} = \frac{6y}{2} = 3y$$
Thus,
$$p - a = 3y - (2y - 1) = 3y - 2y + 1 = y + 1,$$
$$p - b = 3y - 2y = y,$$
$$p - c = 3y - 2y + 1 = y - 1.$$

By Theorem 3.1.2,
$$A = \sqrt{p(p - a)(p - b)(p - c)}$$
$$A = \sqrt{3y(y + 1)(y)(y - 1)} = \sqrt{3y^2(y^2 - 1)}$$
$$A = y\sqrt{3(y^2 - 1)}\ .$$

Thus for A to be integer, $\sqrt{3(y^2 - 1)}$ must be an integer. Hence, $\sqrt{3(y^2 - 1)} = \sqrt{t^2}$, where $t = 3(y^2 - 1)$. That is, $A = y\sqrt{t^2} = yt$. Therefore, ΔABC is Heron if $3(y^2 - 1) = t^2$. □

Theorem 3.2.7 A CH-triangle ABC has sides $2y_n - 1, 2y_n$, and $2y_n + 1$.

Proof : From the proof of Theorem 3.2.6, $3(y^2 - 1) = t^2$ where y is a positive integer. Thus, the prime $3|t^2$. It follows that $t^2 = 3k$, for some integer k. Hence, $k = 3^i$ for some integer $i \geq 1$ and i is odd. This implies that
$$t^2 = 3 \cdot 3^i = 3^{i+1} = 3^2 \cdot 3^{i-1} = 3^2 \cdot 3^{2l+1-1},$$
for some $l \in Z$. Hence,
$$t = 3\sqrt{3^{2l}} = 3 \cdot 3^l\ .$$
Thus, $3|t$. Let $t = 3u$. Then

$$3(y^2 - 1) = t^2$$
$$3y^2 - 3 = (3u)^2$$
$$3y^2 - 3 = 9u^2$$
$$\frac{3y^2}{3} - \frac{3}{3} = \frac{9u^2}{3}$$
$$y^2 - 1 = 3u^2$$

$$y^2 - 3u^2 = 1$$
$$y^2 - \sqrt{3}u^2 = 1$$
$$(y - \sqrt{3}u)(y + \sqrt{3}u) = 1.$$

By Theorem 2.1.19, taking $x_n = y_{n+1},\ y_n = u_{n+1}$, $x_1 = 2$ and $y_1 = 1$,

$$\begin{aligned} y_{n+1} + u_{n+1}\sqrt{3} &= \left(2 + \sqrt{3}\right)^{n+1} \\ &= (2 + \sqrt{3})^n(2 + \sqrt{3}) \\ &= (y_n + u_n\sqrt{3})(2 + \sqrt{3}) \\ &= 2\,y_n + \sqrt{3}\,y_n + 2\sqrt{3}\,u_n + 3\,u_n \\ &= (2\,y_n + 3\,u_n) + \left(\sqrt{3}\,y_n + 2\sqrt{3}\,u_n\right) \\ &= (2\,y_n + 3\,u_n) + \sqrt{3}(y_n + u_n). \end{aligned}$$

Thus, the recurrence relations

$$\begin{cases} y_{n+1} = 2y_n + 3u_n \\ u_{n+1} = y_n + 2u_n \end{cases} \quad (\mathrm{n} = 1, 2, 3, \ldots)$$

give all solutions of $y^2 - 3u^2 = 1$. That is, all CH-triangles have sides $2y_n - 1, 2y_n$, and $2y_n + 1$.

Illustration 3.2.8 For $(y_1, u_1) = (2, 1)$:

$a = 2y_1 - 1 = 2(2) - 1 = 3,$
$b = 2y_1 = 2(2) = 4,$
$c = 2y_1 + 1 = 2(2) + 1 = 5;$

and

$y_2 = 2y_1 + 3u_1 = 2(2) + 3(1) = 7,$
$u_2 = y_1 + 2u_1 = 2 + 2(1) = 4.$

Therefore, $(y_2, u_2) = (7, 4)$.

For $(y_2, u_2) = (7,4)$:

$a = 2y_2 - 1 = 2(7) - 1 = 13,$
$b = 2y_2 = 2(7) = 14,$
$c = 2y_2 + 1 = 2(7) + 1 = 15;$

and

$y_3 = 2y_2 + 3u_2 = 2(7) + 3(4) = 26,$
$u_3 = y_2 + 2u_2 = 7 + 2(4) = 15.$

Therefore, $(y_3, u_3) = (26, 15)$.

For $(y_3, u_3) = (26,15)$:

$a = 2y_3 - 1 = 2(26) - 1 = 51,$
$b = 2y_3 = 2(26) = 52,$
$c = 2y_3 + 1 = 2(26) + 1 = 53;$

and

$y_4 = 2y_3 + 3u_3 = 2(26) + 3(15) = 97,$
$u_4 = y_3 + 2u_3 = 26 + 2(15) = 56.$

Therefore, $(y_4, u_4) = (97, 56)$.

For $(y_4, u_4) = (97, 56)$:

$a = 2y_4 - 1 = 2(97) - 1 = 193,$
$b = 2y_4 = 2(97) = 194,$
$c = 2y_4 + 1 = 2(97) + 1 = 195;$

By $y_4 = 2,7,26,97, \ldots$ we get the CH-triangles $(3, 4, 5)$; $(13, 14, 15)$; $(51, 52, 53)$; $(193, 194, 195)$; $\ldots$.
The ΔABC is Figure 3.4 is also a CH-triangle.

Corollary 3.2.9 In a CH-triangle with sides of lengths $2y - 1$, $2y$, and $2y + 1$, the inradius r is always an integer.

Proof : Let r be the inradius, A denotes the area and p is the semiperimeter of the CH-triangle. Then $r = \frac{A}{p}$. By Theorem 3.2.6, $p = 3y$ and $A = yt$. Thus,

$$r = \frac{A}{p} = \frac{A}{3y} = \frac{yt}{3y} = \frac{t}{3}.$$

From the proof of Theorem 3.2.7, $t = 3u$. Hence,

$$r = \frac{t}{3} = \frac{3u}{3} = u$$

Therefore, r is an integer. □

Corollary 3.2.10 In a CH-triangle with sides of lengths $2y - 1$, $2y$, and $2y + 1$, the height h_{2y} denotes the height corresponding to the (single) even side is an integer.

Proof : Let A denotes the area and h_{2y} the height corresponding to the side $2y$. Then $A = \frac{1}{2}(2y)h_{2y}$. Thus, $h_{2y} = \frac{2A}{2y} = \frac{A}{y}$.

From the proof of Theorem 3.2.7, $t = 3u$, where $u = r$ by the proof of Corollary 3.2.9. Hence, $h_{2y} = \frac{A}{y} = \frac{yt}{y} = t = 3u = 3r$. Therefore, h_{2y} is an integer. □

Corollary 3.2.11 In a CH-triangle, which is not a P-triangle (that is, excluding the triangle (3,4,5), all other heights cannot be integers.
Proof : For side $2y - 1$, let x be the height. Then

$$yt = A = \frac{(2y - 1)x}{2}$$
$$2yt = (2y - 1)x$$
$$x = \frac{2yt}{2y-1}.$$

Since $(2y - 1, 2y) = 1$, so $x = \frac{2yt}{2y-1}$ is integer only if $(2y - 1)|t$ where $t = 3u$ from the proof of Theorem 3.2.7. That is, $(2y - 1)|3u$.

Now $(2y - 1)|3u$ implies $(2y - 1)|3u^2 = y^2 + 1$ since $y^2 - 3u^2 = 1$ from the proof of Theorem 3.2.7. Thus, $(2y - 1)|y^2 - 1$. □

Corollary 3.2.12 Let R denotes the radius of the circumscribed circle of CH-triangle with sides a, b, c, then it is not an integer.

Proof : Note that

$$R = \frac{abc}{4A} = \frac{2y(2y + 1)(2y - 1)}{4yt} = \frac{2y(4y^2 - 1)}{4yt}$$
$$R = \frac{4y^2-1}{2t} = \frac{\text{odd}}{\text{even}} \neq \text{integer}.$$

Therefore, the result follows. □

Corollary 3.2.13 Let r_{2y} denotes the radius of the exscribed circle corresponding to the side of length $2y$. Then r_{2y} is an integer.

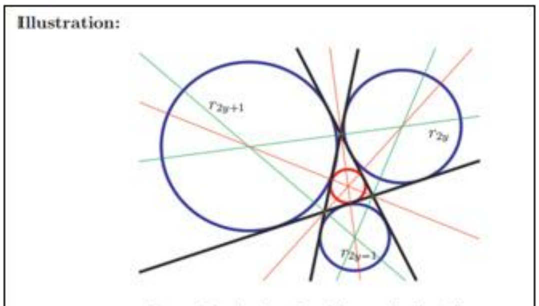

Figure 3.5: A triangle with exscribed circles

Proof: Now, $r_{2y} = \frac{A}{p-2y} = \frac{yt}{3y-2y} = \frac{yt}{y} = t$. Hence, r_{2y} is integer.

Corollary 3.2.14 Let r_{2y-1} denote the radius of the exscribed circle corresponding to the side of length $2y-1$. Then r_{2y-1} is integer only in the P-triangle (3,4,5).

Proof : Now,

$$r_{2y-1} = \frac{A}{p-(2y-1)} = \frac{yt}{3y-(2y-1)} = \frac{yt}{\mathrm{p}-2\mathrm{y}+1}$$
$$r_{2y-1} = \frac{yt}{\mathrm{y}+1}.$$

Since $(y+1, y) = 1$, so r_{2y-1} is integer only when $(y+1)|t$ where $t = 3u$, and since $y^2 - 3u^2 = 1$ then

$$y^2 - 1 = 3u^2$$
$$(y-1)(y+1) = u(3u).$$

Now $(y+1)|3u$ implies that there exists $k \in Z$ such that $3u = (y+1)k$. Thus

$$(y-1)(y+1) = u(y+1)k$$
$$3(\mathrm{y}-1)(\mathrm{y}+1) = 3\mathrm{u}(\mathrm{y}+1)\mathrm{k}$$
$$(y-1)(y+1) = u(y+1)k$$
$$y-1 = uk$$
$$3(y-1) = 3uk.$$

Substituting $3u = (y+1)k$, we have

$$3(y-1) = 3uk$$
$$3(y-1) = [(y+1)k]k$$
$$k^2 = \frac{3(y-1)}{y+1} = \frac{3y-3}{y+1} = \frac{3y-3+3-3}{y+1}$$
$$k^2 = \frac{3y+3-6}{y+1} = \frac{3(y+1)}{y+1} - \frac{6}{y+1}$$
$$k^2 = 3 - \frac{6}{y+1}$$

Now, $(y+1)|6$ when $\mathrm{y} \in \{1, 2, 5\}$. But the only value of y is 2 when $k = 1$. Hence, when $y = 2$, the CH-triangle has sides 3, 4, and 5 which is a P-triangle.

Corollary 3.2.15 Let r_{2y+1} denote the radius of the exscribed circle corresponding to the side of length $2y+1$. Then r_{2y+1} is integer in the P-triangle (3, 4, 5) and P-triangle (13, 14, 15).

Proof : Now,

$$r_{2y+1} = \frac{A}{p-(2y+1)} = \frac{yt}{3y-(2y+1)}$$
$$r_{2y+1} = \frac{yt}{3y-2y-1} = \frac{yt}{y-1}.$$

Since $(y-1, y) = 1$, so r_{2y+1} is integer only when $(y-1)|t$ where $t = 3u$. Since $y^2 - 3u^2 = 1$ then

$$y^2 - 1 = 3u^2$$
$$(y-1)(y+1) = u(3u).$$

Now $(y-1)|3u$ implies that there exists $k \in Z$ such that $3u = (y-1)k$. Thus

$$(y-1)(y+1) = u(y-1)k$$
$$y+1 = uk$$
$$3(y+1) = 3uk.$$

Substituting $3u = (y-1)k$, gives

$$3(y+1) = 3uk$$
$$3(y+1) = [(y-1)k]k$$
$$k^2 = \frac{3(y+1)}{y-1} = \frac{3y+3}{y-1} = \frac{3y+3-3+3}{y-1}$$
$$k^2 = \frac{3y-3+6}{y-1} = \frac{3(y-1)+6}{y-1}$$
$$k^2 = \frac{3(y-1)}{y-1} + \frac{6}{y-1} = 3 + \frac{6}{y-1}.$$

Thus, $(y-1)|6$ when $y \in \{2, 3, 4, 7\}$. But $y = 7$ or 2 when $k = 2$ or 3, respectively. Hence, when $y = 2$, the CH-triangle has sides 3, 4, and 5 which is a P-triangle and when $y = 7$, the CH-triangle has sides 13, 14, and 15 which is also a P-triangle.

3.2.3 Isosceles Triangle

Theorem 3.2.16 Let ABC be an isosceles triangle with $|AB| = |AC| = b$, $|BC| = a$, $|AA'| = x$ and $|BB'| = y$ are integers. Then triangle ABC is a Heron triangle.

Illustration:

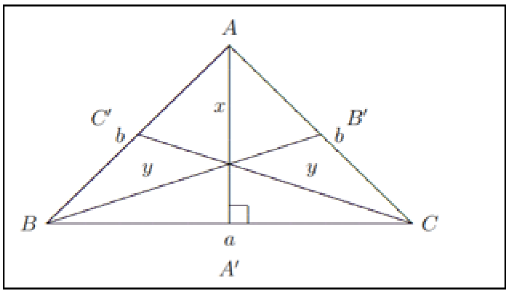

Figure 3.6: An isosceles ΔABC

Proof: From the Pythagorean Theorem,

$$b^2 = x^2 + \left(\frac{a}{2}\right)^2 = x^2 + \frac{a^2}{4},$$

$b^2 - x^2 = \frac{a^2}{4}$ is an integer. Implying that a is even.

Let a be an even number, that is, $a = 2u$, where u is an integer. Then

$$b^2 = x^2 + \left(\frac{a}{2}\right)^2 = x^2 + \frac{(2u)^2}{4} = x^2 + \frac{4u^2}{4} = x^2 + u^2.$$

Since x is an integer and $a = 2u$, so the sides of ΔABC are integers. Also, since

$$A = \frac{1}{2}bh = \frac{1}{2}ax = \frac{1}{2}(2u)x = xu.$$

Therefore, ΔABC is H-triangle. □

Remark 3.2.17 The general solutions of $b^2 = x^2 + u^2$ can be written as one of the following:

(i) $b = \lambda(m^2 + n^2), x = \lambda(m^2 - n^2), u = 2\lambda mn$; or
(ii) $b = \lambda(m^2 + n^2), x = 2\lambda mn, u = \lambda(m^2 - n^2)$.

Proof : It follows from Theorem 2.1.18 and Lemma 3.1.4. □

Note that from Remark 3.2.17,

(i) $x^2 + u^2 = (m^2 - n^2)^2 + 4\lambda^2 m^2 n^2$
$= \lambda^2(m^2 - 2m^2n^2 + n^2) + 4\lambda^2 m^2 n^2$
$= \lambda^2(m^2 - 2m^2n^2 + n^2 + 4m^2n^2)$
$= \lambda^2(m^2 + 2m^2n^2 + n^2)$
$= \lambda^2(m+n)^2 = b^2.$

Similarly, (ii) can be easily verified.

Corollary 3.2.18 In an isosceles H-triangle having all heights which are integers and a is the base of the triangle,
(i) $a = 4smn(m^2 + n^2)$ and $b = s(m^2 + n^2)^2$ or
(ii) $a = 2s(m^4 - n^4)$ and $b = s(m^2 + n^2)^2$.

Proof : (i) From the proof of Theorem 3.2.16, $a = 2u$. By Remark 3.2.17(i),
$a = 2u = 2(2\lambda mn) = 4\lambda mn, and\ b = \lambda(m^2 + n^2)$;
$A = ux = \lambda(m^2 - n^2)(2\lambda mn) = 2\lambda^2 mn(m^2 - n^2)$.

Now, $A = \frac{bh}{2} = \frac{by}{2}$. Thus,
$y = \frac{2A}{b} = \frac{2[2\lambda^2 mn(m^2 - n^2)]}{\lambda(m^2 + n^2)} = \frac{4\lambda^2 mn(m^2 - n^2)}{\lambda(m^2 + n^2)}$,
where y is an integer only when $\lambda(m^2 + n^2)|4\lambda^2 \mathrm{mn}(m^2 - n^2)$. Since $(m^2 + n^2, 4mn(m^2 - n^2)) = 1$, by the remarks in Section 2.2, it follows that $(m^2 + n^2)|\lambda$, that is, $\lambda = s(m^2 + n^2)$ for some $s \in Z$. Hence,
$a = 2u = 2[2\lambda mn] = 4\lambda mn$
$= 4[s(m^2 + n^2)](mn) = 4smn((m^2 + n^2)$;
$b = \lambda(m^2 + n^2) = s(m^2 + n^2)](m^2 + n^2)$
$= s(m^2 + n^2)^2$.

Similarly for (ii) and with $a = 2u$, apply Remark 3.2.17(ii), so that
$a = 2u = 2\lambda(m^2 - n^2)\ and\ b = \lambda(m^2 + n^2)$;
$A = ux = \lambda(m^2 - n^2)(2\lambda mn) = 2\lambda^2 mn((m^2 - n^2)$.

Now, $A = \frac{bh}{2} = \frac{by}{2}$. Thus,
$y = \frac{2A}{b} = \frac{2[2\lambda^2 mn(m^2 - n^2)]}{\lambda(m^2 + n^2)} = \frac{4\lambda^2 mn(m^2 - n^2)}{\lambda(m^2 + n^2)}$,
where y is an integer only when $\lambda(m^2 + n^2)|4\lambda^2 \mathrm{mn}(m^2 - n^2)$.Since $(m^2 + n^2, 4mn(m^2 - n^2)) = 1$, it implies that $(m^2 + n^2)|\lambda$, that is, $\lambda = s(m^2 + n^2)$for some $s \in Z$. Hence,
$A = 2u = 2[\lambda(m^2 - n^2)] = 2\lambda(m^2 - n^2)$
$= 2[s(m^2 + n^2)](m^2 - n^2) = 2s(m^4 - n^4)$;
$B = \lambda(m^2 + n^2) = [s(m^2 + n^2)](m^2 + n^2)$
$= s(m^2 + n^2)^2$.
Therefore, the result follows.

Corollary 3.2.19 If an isosceles ΔABC with integer sides a, c, b (base a) is an H-triangle, then its height x must be an integer.

Proof : Let
$p = \frac{a+b+c}{2} = \frac{a+b+b}{2}$ (since $c = b$)
$p = \frac{a + 2b}{2} = \frac{a}{2} + b$.
Thus, p is an integer, when a is even. Let $a = 2u$, for some integer u. Then
$p = b + u$,
$p - a = b + u - a = b + u - 2u = b - u$,
$p - b = p - c = b + u - b = u$.
It follows that
$A = \sqrt{p(p-a)(p-b)(p-c)} = \sqrt{p(p-a)(p-b)^2}$
$= \sqrt{(b+u)(b+-u)u^2} = u\sqrt{b^2 - u^2}$.
This is integer only when $b^2 - u^2 = q^2$, that is, $A = uq$. Now $b^2 - u^2$ is in fact x^2 (where x is the height corresponding to the base a), so $q = x$. In other words, if an isosceles triangle ABC is H-triangle then its height x must be integer. □

Corollary 3.2.20 In an isosceles H-triangle, r is integer only when
(i) $b = s(m + n)(m^2 + n^2)$ and $a = 4mns(m + n)$ or
(ii) $b = sm(m^2 + n^2)$ and $a = 2sm(m^2 - n^2)$.

Proof : (i) Note that $r = \frac{A}{p} = \frac{uq}{b+u}$, where $b^2 = u^2 + q^2$. Applying Remark 3.2.17(i), then write the following equations
$b = \lambda(m^2 + n^2), u = 2\lambda mn, q = \lambda(m^2 - n^2)$,
$p = b + u = \lambda(m^2 + n^2) + 2\lambda mn$
$= \lambda(m^2 + n^2 + 2mn) = \lambda(m^2 + 2mn + n^2)$
$= \lambda(m + n)^2$,
$A = uq = 2\lambda mn[\lambda(m^2 - n^2)] = 2\lambda^2 mn(m^2 - n^2)$.

Now, r is an integer if $b + u|uq$, that is, $\lambda(m + n)^2|2\lambda^2 mn(m^2 - n^2)$ implies $(m + n)^2|2\lambda mn(m^2 - n^2)$ implies $(m + n)|2\lambda mn(m - n)$. Since $(m + n, 2mn(m - n)) = 1$, then it follows that $(m + n)|\lambda$, that is, $\lambda = s(m + n)$ for some $s \in Z$. Hence,
$b = \lambda(m^2 + n^2) = s(m + n)(m^2 + n^2)$ and
$a = 2u = 2(2\lambda mn) = 4\lambda mn$
$= 4[s(m + n)](mn) = 4mns(m + n)$.

Similarly for (ii) applying Remark 3.2.17(ii) and write the following equations
$b = \lambda(m^2 + n^2), u = \lambda(m^2 - n^2), q = 2\lambda mn$,
$p = b + u = \lambda(m^2 + n^2) + \lambda(m^2 - n^2)$
$= \lambda(m^2 + n^2 + m^2 - n^2) = 2\lambda m^2$,
$A = uq = \lambda(m^2 - n^2)(2\lambda mn) = 2\lambda^2 mn(m^2 - n^2)$.

Now, r is an integer if $b + u|uq$, that is, $2\lambda m^2|2\lambda^2 mn(m^2 - n^2)$ which implies $m^2|\lambda mn(m^2 - n^2)$and further implies, $m|\lambda n(m^2 - n^2)$. Since $(m, n) = 1$, then it implies that $m|\lambda$, that is $\lambda = sm$ for some $s \in Z$. Hence,
$b = \lambda(m^2 + n^2) = sm(m^2 + n^2)$ and
$a = 2u = 2\lambda(m^2 - n^2) = 2sm(m - n)$.
Therefore, the result follows.

Corollary 3.2.21 In an isosceles H-triangle with sides a, b, c where $b = c$, R is integer when
(i) $\lambda = 2s(m^2 - n^2)$ and
(ii) $\lambda = 4mns$.

Proof : (i) Note that $2A = xu$ from the proof of Theorem 3.2.16. Then
$R = \frac{1}{2}c = \frac{abc}{4A} = \frac{ab^2}{4A} = \frac{2ub^2}{4uq} = \frac{b^2}{2q}$.
R is an integer only when $2q|b^2$, where $b^2 = u^2 + q^2$. Applying Remark 3.2.17(i) we get $2q|b^2$, that is, $2[\lambda(m^2 - n^2)]|[\lambda(m^2 + n^2)]^2$ which implies $2\lambda(m^2 - n^2)|\lambda^2(m^2 + n^2)^2$ and gives $2(m^2 - n^2)|\lambda(m^2 + n^2)^2$. Since $(2(m^2 - n^2), (m^2 + n^2)^2) = 1$, then it follows that $2(m^2 - n^2)|\lambda$, that is, $\lambda = 2s(m^2 - n^2)$ for some $s \in Z$.

Similarly for (ii), apply Remark 3.2.17(ii) and get $2q|b^2$, that is, $2[2\lambda mn]|[\lambda(m^2 + n^2)]^2$ which implies $4\lambda mn|\lambda^2(m^2 + n^2)^2$ and gives $4mn|\lambda(m^2 + n^2)$. Since $(4mn, m^2 + n^2) = 1$, then it implies that $4mn|\lambda$, that is, $\lambda = mns$ for some $s \in Z$.

Corollary 3.2.22 Let r_a denotes the radius of the exscribed circle corresponding to side of length a. Then r_a is integer only when
(i) $\lambda = s(m-n)$ and
(ii) $\lambda = sn$.

Proof : Note that

$$r_a = \frac{A}{p-a} = \frac{uq}{b-u}$$

where $b^2 = u^2 + q^2$.

Applying Remark 3.2.17(i)

$$A = uq = (2\lambda mn)[\lambda(m^2-n^2)] = 2\lambda^2 mn(m^2-n^2),$$
$$p-a = b-u = \lambda(m^2+n^2) - 2\lambda mn$$
$$= \lambda[(m^2+n^2) - 2mn]$$
$$= \lambda[m^2 - 2\lambda mn + n^2] = \lambda(m-n)^2.$$

Thus,

$$r_a = \frac{uq}{b-u} = \frac{2\lambda^2 mn(m^2-n^2)}{\lambda(m-n)^2},$$

that is, $\lambda(m-n)^2 | 2\lambda^2 mn(m^2-n^2)$ which implies $(m^2-n^2)|2\lambda mn(m+n)$. Since $(m-n, 2mn(m+n)) = 1$, then it implies that $(m-n)|\lambda$, that is, $\lambda = s(m-n)$, for some $s \in Z$.

Similarly for (ii), applying Remark 3.2.17(ii)

$$A = uq = (2\lambda mn)[\lambda(m^2-n^2)] = 2\lambda^2 mn(m^2-n^2),$$
$$p-a = b-u = \lambda(m^2+n^2) - \lambda(m^2-n^2)$$
$$= \lambda[(m^2+n^2) - (m^2-n^2)]$$
$$= \lambda[m^2+n^2 - m^2+n^2]$$
$$= \lambda(2n^2) = 2\lambda n^2.$$

Thus,

$$r_a = \frac{uq}{b-u} = \frac{2\lambda^2 mn(m^2-n^2)}{2\lambda n^2} = \frac{\lambda m(m^2-n^2)}{n} = \frac{\lambda m^3}{n} - \lambda\text{mn}.$$

Hence, $n|\lambda m^3$. Since $(m,n) = 1$, then it follows that $n|\lambda$, that is, $\lambda = sn$, for some $s \in Z$. Therefore, r_a is integer only if in (i) $\lambda = s(m-n)$ and in (ii) $\lambda = sn$. □

3.2.4 Integral Triangle

Theorem 3.2.23 An integral triangle of sides a, b, c is H-triangle if a, b, c can be represented in the following forms

$$a = \frac{(m-n)(k^2+mn)}{d},\ b = \frac{m(k^2+n^2)}{d},\ c = \frac{n(k^2+m^2)}{d}$$

where d, m, n, k are positive integers; $m > n$; and d is an arbitrary common divisor of $(m-n)(k^2+mn), m(k^2+n^2)$, and $n(k^2+m^2)$.

Proof : Let

$$a = \frac{(m-n)(k^2+mn)}{d},\ b = \frac{m(k^2+n^2)}{d},\ c = \frac{n(k^2+m^2)}{d},$$

where d, m, n, k are positive integers; $m > n$; and d is an arbitrary common divisor of $(m-n)(k^2+mn), m(k^2+n^2)$, and $n(k^2+m^2)$.

Calculating p and A:

$$p = \frac{\frac{(m-n)(k^2+mn)}{d} + \frac{m(k^2+n^2)}{d} + \frac{n(k^2+m^2)}{d}}{2}$$
$$p = \frac{(m-n)(k^2+mn) + m(k^2+n^2) + n(k^2+m^2)}{2d}$$
$$p = \frac{mk^2 - nk^2 + m^2n - mn^2 + mk^2 + mn^2 + nk^2 + m^2n}{2d}$$
$$p = \frac{mk^2 + mk^2 - nk^2 + nk^2 + m^2n + m^2n + mn^2 - mn^2}{2d}$$
$$p = \frac{2mk^2 + 2m^2n}{2d}$$
$$p = \frac{2m(k^2+mn)}{2d}$$
$$p = \frac{m(k^2+mn)}{d}$$

and

$$p-a = \frac{m(k^2+mn)}{d} - \frac{(m-n)(k^2+mn)}{d}$$
$$p-a = \frac{k^2m + m^2n - (k^2m - k^2n + m^2n - mn^2)}{d}$$
$$p-a = \frac{k^2m + m^2n - k^2m + k^2n - m^2n + mn^2}{d}$$
$$p-a = \frac{k^2n + mn^2}{d}$$
$$p-a = \frac{n(k^2+mn)}{d},$$

$$p-b = \frac{m(k^2+mn)}{d} - \frac{(m-n)(k^2+n^2)}{d}$$
$$p-b = \frac{k^2m + m^2n - k^2m - mn^2}{d}$$
$$p-b = \frac{m^2n - mn^2}{d}$$
$$p-b = \frac{mn(m-n)}{d},$$

$$p-c = \frac{m(k^2+mn)}{d} - \frac{n(k^2+m^2)}{d}$$
$$p-c = \frac{k^2m + m^2n - k^2n - m^2n}{d}$$
$$p-c = \frac{k^2m - k^2n}{d}$$
$$p-c = \frac{k^2(m-n)}{d}.$$

Hence,

$$p(p-a)(p-b)(p-c)$$
$$= \frac{m(k^2+mn)}{d} \cdot \frac{n(k^2+mn)}{d} \cdot \frac{mn(m-n)}{d} \cdot \frac{k^2(m-n)}{d}$$
$$= \frac{k^2m^2n^2(k^2+mn)^2(m-n)^2}{d^4}.$$

Thus,

$$A = \sqrt{p(p-a)(p-b)(p-c}$$
$$A = \sqrt{\frac{k^2m^2n^2(k^2+mn)^2(m-n)^2}{d^4}}$$
$$A = \frac{kmn(k^2+mn)(m-n)}{d^2}$$

Hence, p and A are integers for all k, m, n, d as given above. Therefore, the integral triangle is H-triangle.

4. Summary and Recommendations

This chapter presents the summary of the results generated in this study. Moreover, some recommendations are given for further investigation.

4.1 Summary

This study successfully classify and examines Heron triangles, highlighting their unique mathematical properties and behaviors. By presenting rigorous and proof and examples, it opens avenues for further exploration, including generalized Heron triangles and their applications in advanced geometry.

The following are the results in this study:

1) If a triangle has sides of lengths a, b, c and if $p = \frac{1}{2}(a + b + c)$, the "semiperimeter" of the triangle, then Heron's formula says that the area A of the triangle is given by $A = \sqrt{p(p - a)(p - b)(p - c)}$ (Theorem3.1.2)

2) The general solution of the equation $a^2 + b^2 = c^2$ are given by $a = \lambda(m^2 - n^2), b = 2\lambda mn \text{ and } c = \lambda(m^2 + n^2)$ (if b is even) where λ is an arbitrary positive integer, while m > n are relatively prime of different parities (that is, (m,n) =1 and m and n cannot be both odd or even). (Lemma 3.1.4)

3) Let $a = \lambda(m^2 - n^2), b = 2\lambda mn$ and $c = \lambda(m^2 + n^2)$ be the length of the sides of a P-triangle ABC where AB is the hypotenuse and λ is an arbitrary positive integer, while m > n are relatively prime of different parities (that is, (m, n) = 1 and m and n cannot be both odd or even). Then ΔABC is a Heron triangle. (Theorem 3.2.1)

4) Let p be the semiperimeter and r be the inradius of a Heron triangle where $p = \frac{a+b+c}{2}, r = p - c$, then p and r are integers. (Corollary 3.2.2)

5) Let R be the radius of the circumscribed circle and h_a, h_b, h_c be the heights of a Heron ΔABC in Figure 3.3, where $R = \frac{1}{2}c$ and $h_a = \frac{2A}{a}$, $h_b = \frac{2A}{b}, h_c = \frac{2A}{c}$. Then R is an integer if λ is even and the heights h_a, h_b, h_c are all integers if $c|ab$. (Corollary 3.2.3)

6) In a P-triangle with sides a, b, and c, the quantities $A, h_a, h_b, h_c, r,$ and R are integers at the same time if and only if a, b, and c, are given by $a = 2d(m^4 - n^4), b = 4dmn(m^2 + n^2), c = 2d(m^2 + n^2)^2$ where λ is even, $(m, n) = 1$, and m and n with m > n are of different parities. (Theorem 3.2.4)

7) Let $2y - 1, 2y, 2y + 1$ be the sides of a triangle ABC where y is a positive integer. Then ΔABC is a Heron triangle specifically known as CH-triangle. (Theorem 3.2.6)

8) A CH-triangle ABC has sides $2y_n - 1, 2y_n$, and $2y_n + 1$. (Theorem 3.2.7)

9) In a CH-triangle with sides of lengths $2y - 1, 2y, 2y + 1$, the inradius r is always an integer. (Corollary 3.2.9)

10) In a CH-triangle with sides of lengths $2y - 1, 2y$, and $2y + 1$, the height h_{2y} denotes the height corresponding to the (single) even side is an integer. (Corollary 3.2.10)

11) In a CH-triangle, which is not a P-triangle (that is, excluding the triangle (3,4,5), all other heights cannot be integers. (Corollary 3.2.11)

12) Let R denotes the radius of the circumscribed circle of CH-triangle with sides a, b, c, then it is not an integer. (Corollary 3.2.12)

13) Let r_{2y} denotes the radius of the exscribed circle corresponding to the side of length 2y. Then r_{2y} is an integer. (Corollary 3.2.13)

14) Let r_{2y-1} denote the radius of the exscribed circle corresponding to the ide of length $2y - 1$. Then r_{2y-1} is integer only in the P-triangle (3, 4, 5). (Corollary 3.2.14)

15) Let r_{2y+1} denote the radius of the exscribed circle corresponding to the side of length 2y + 1. Then r_{2y+1} is integer in the P-triangle (3,4,5) and P-triangle (13,14,15). (Corollary 3.2.15)

16) Let ABC be an isosceles triangle with $|AB| = |AC| = b$, $|BC| = a$, $|AA^1| = x$ and $|BB^1| = y$ are integers. Then triangle ABC is a Heron triangle. (Theorem 3.2.16)

17) In an isosceles H-triangle having all heights which are integers and a is the base of the triangle,
a) $a = 4smn(m^2 + n^2)$ and $b = s(m^2 + n^2)^2$ or
b) $a = 2s(m^4 - n^4)$ and $b = s(m^2 + n^2)^2$.
(Corollary 3.2.18)

18) If an isosceles ΔABC with integer sides *a*, *c*, *b* (base *a*) is H-triangle, then its height x must be an integer.
(Corollary 3.2.19)

19) In an isosceles *H*-triangle, *r* is integer only when
(i) $b = s(m + n)(m^2 + n^2)$ and $a = 4mns(m + n)$ or
(ii) $b = sm(m^2 + n^2)$ and $a = 2sm(m^2 - n^2)$.
(Corollary 3.2.20)

20) In an isosceles H-triangle with sides *a*, *b*, *c* where *b* = *c*, *R* is integer when
(i) $\lambda = 2s(m^2 - n^2)$ and
(ii) $\lambda = 4mns$. (Corollary 3.2.21)

21) Let r_a denotes the radius of the exscribed circle corresponding to side of length a. Then r_a is integer only when
(i) $\lambda = s(m - n)$ and
(ii) $\lambda = sn$. (Corollary 3.2.22)

22) An integral triangle of sides *a*, *b*, *c* is *H*-triangle if *a*, *b*, *c* can be represented in the following forms:
$a = \frac{(m-n)(k^2+mn)}{d}, b = \frac{m(k^2+n^2)}{d}, c = \frac{n(k^2+m^2)}{d}$,
where *d*, *m*, *n*, *k* are positive integers; *m* > *n*; and *d* is an arbitrary common divisor of $(m - n)(k^2 + mn)$, $m(k^2 + n^2)$, and $n(k^2 + m^2)$. (Theorem 3.2.23)

4.2 Recommendations

The following are recommended for further study:
1) Other types of Heron triangle and its characterization.
2) Characterization of a Generalized Heron triangle.

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