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Topological Polynomials and Indices of Line Graphs of Wheel Graphs

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Abstract: ***First Zagreb polynomial of a graph G with vertex set V(G) and edge set E(G) is defined as $M_1(G,x) = \sum_{uv\in E(G)} x^{d_u+d_v}$ and the first Zagreb index can be obtained from its polynomial as $M_1(G) = \frac{\partial M_1(G,x)}{\partial x}|_{x=1}$. In this paper some topological polynomials and their indices are obtained for line graph of wheel graph.***

Keywords: Hyper-index, hyper-polynomial, leap degree, line graph of wheel graph, M-polynomial, NM-polynomial, Revan degree, reverse degree, Zagreb index

1. Introduction

Let G be a simple, finite, connected graph with vertex set V(G) and edge set E(G). The degree of a vertex $u \in V(G)$ is denoted by d_u and is the number of vertices adjacent to u. The edge connecting the vertices u and v is denoted by uv. A molecular graph is representation of the structural formula of a chemical compound in terms of graph theory whose vertices correspond to the atoms of compound and edges correspond to chemical bonds. A topological index is a numerical parameter mathematically derived from the graph structure; several such topological indices have been considered in theoretical chemistry and have found some applications in QSPR/QSAR study.

The distance-counting polynomials were studied for titanium dioxide nanotubes in [1].The k-distance degree-based topological indices of molecular graphs were defined and computed in [2-4].First and second neighborhood Gourava indices using NM-polynomials for drug structures were investigated in [5].Many topological polynomials and indices were computed in many papers for example [6-16].Sum degree-based topological indices of nanotubes were computed in [17].Leap reduced reciprocal Randic and leap reduced second Zagreb indices of some graphs were delved by F.Dayan et. al. [18]. Neighborhood degree-based topological indices for some graphs were studied in [19-20]. Closed and open neighborhood of a vertex are useful in discussing the degree of vertices and local properties of graphs. Open neighborhood of a vertex v, denoted by N(v) is the set of vertices that are adjacent to v, excluding itself, i.e. $N(v) = u \in V|(v,u) \in E$ and $\deg(v) = |N(v)|$. Closed neighborhood of a vertex v is denoted by N[v] is the set of vertices that are adjacent to v, including v itself i.e. $N[v] = v\cup N(v)$.

A wheel graph is a type of graph that consists of a central vertex connected to all vertices of a cycle. Wheel graphs are denoted by W_n, where n is the number of vertices in the cycle plus one for the central vertex. The wheel graph W_n with n+1, vertices are defined as the joining of K_1 and C_n, where K_1 is the complete graph with one vertex and C_n is the cycle graph with n vertices. The degree of the central vertex in a wheel graph is n, while each vertex in the cycle has a degree of 3.Line graph of the subdivision graph of wheel graph denoted by $L(S(W_n))$ has order 4n and size $\frac{n^2+9n}{2}$.The diameter of line graph of subdivision graph of wheel graph is 1 for $n = 4$ and 2 for $n \geq 5$. In a graph of $L(S(W_n))$ there are 3n vertices of degree 3 and remaining n vertices of degree n [21-27].

The first, second and hyper reverse Zagreb polynomials [28-29] are defined as;

$$CM_1(G,x) = \sum_{uv\in E(G)} x^{(c_u+c_v)} \quad (1)$$
$$CM_2(G,x) = \sum_{uv\in E(G)} x^{(c_u\times c_v)} \quad (2)$$
$$CHM_1(G,x) = \sum_{uv\in E(G)} x^{(c_u+c_v)^2} \quad (3)$$
$$CHM_2(G,x) = \sum_{uv\in E(G)} x^{(c_u\times c_v)^2} \quad (4)$$

Where the reverse degree of a vertex v is $c_v = \Delta(G) - d_G(v) + 1$.

The first, second and hyper Revan polynomials [30] are defined as;

$$R_1(G,x) = \sum_{uv\in E(G)} x^{(r_u+r_v)}. \quad (5)$$
$$R_2(G,x) = \sum_{uv\in E(G)} x^{(r_u\times r_v)}. \quad (6)$$
$$HR_1(G,x) = \sum_{uv\in E(G)} x^{(r_u+r_v)^2}. \quad (7)$$
$$HR_2(G,x) = \sum_{uv\in E(G)} x^{(r_u\times r_v)^2}. \quad (8)$$

Where Revan degree of a vertex u is $r_G(u) = \Delta(G) + \delta(G) - d_G(u)$.

The first, second and leap hyper Zagreb polynomials are defined as [31-32];

$$LM_1^*(G,x) = \sum_{uv\in E(G)} x^{d_2(u)+d_2(v)} \quad (9)$$
$$LM_2(G,x) = \sum_{uv\in E(G)} x^{d_2(u)\times d_2(v)} \quad (10)$$
$$LHM_1(G,x) = \sum_{uv\in E(G)} x^{[d_2(u)+d_2(v)]^2} \quad (11)$$
$$LHM_2(G,x) = \sum_{uv\in E(G)} x^{[d_2(u)\times d_2(v)]^2} \quad (12)$$

The M-polynomial correspond degree to degree-based indices, while the NM-polynomial parallels this for neighborhood degree-based indices [33-35]. M-polynomials for $M_1(G)$,$M_2(G)$,$HM_1(G)$ and $HM_2(G)$ are defined from the formula of M-polynomial of graph. M-polynomial is defined as

$$M(G;x,y) = \sum_{\delta\le i\le j\le\Delta} m_{ij}(G)\, x^i y^j, \quad (13)$$

where $\delta = \min\{d_v|v\in V(G)\}$, $\Delta = \max\{d_v|v\in V(G)\}$, and $m_{ij}(G)$ is the edge $vu \in E(G)$ such that $i \le j$, with $D_x = x\,\frac{\partial f(x,y)}{\partial x}$, $D_y = y\,\frac{\partial f(x,y)}{\partial y}$, $S_x=\int_0^x \frac{f(t,y)}{t}dt$, $S_y = \int_0^y \frac{f(x,t)}{t}dt$, $J(f(x,y)) = f(x,x)$ and $Q_\alpha(f(x,y)) = x^\alpha (f(x,y))$.

The first, second and hyper Zagreb indices can be computed from M-polynomial as;

$M_1(G)=(D_x + D_y)(M(G;x,y))|_{x=y=1}$.

$M_2(G)=(D_x \times D_y)(M(G;x,y))|_{x=y=1}$.
$HM_1(G)=(D_x + D_y)^2(M(G;x,y))|_{x=y=1}$.
$HM_2(G)=(D_x \times D_y)^2(M(G;x,y))|_{x=y=1}$.

NM-polynomials for $NM_1(G)$, $NM_2(G)$, $NHM_1(G)$ and $NHM_2(G)$ can be defined on open neighborhood $N(v)$ of a vertex.

NM-polynomial of graph G is defined as [36-37];
$NM(G;x,y) = \sum_{i\le j} m_{ij}(G)\, x^i y^j$.

Where m_{ij} is the total number of edges $vu\in E(G)$, such that $\{\delta_u, \delta_v\} = \{i, j\}$ and δ_u, δ_v are used in the definition of neighborhood degree-based indices.

The first Zagreb index can be calculated as derivative of first Zagreb polynomial at $x = 1$ [38],

$$M_1(G) = \frac{\partial M_1(G,x)}{\partial x}\Big|_{x=1}. \quad (14)$$

The edge partition for degree of end vertices in line graph of subdivision graph of wheel graph is;
$E_{(3,3)} = \{uv\in E_G(L(S(W_n)))|d_u=3,d_v=3\}$, $|E_{(3,3)}| = 4n$;
$E_{(3,n)} = \{uv\in E_G(L(S(W_n)))|d_u=3,d_v=n\}$, $|E_{(3,n)}| = n$;
$E_{(n,n)} = \{uv\in E_G(L(S(W_n)))|d_u=n,d_v=n\}$, $|E_{(n,n)}| = \frac{n(n-1)}{2}$.

Symbols and notations used in this paper are standard and mainly taken from standard books of graph theory [39-40]. In this paper reverse, Revan, leap degree-based first, second, hyper first and second Zagreb polynomials, M-polynomials and NM-polynomials and their indices are obtained for line graph of subdivision graph of wheel graph.

2. Materials and Method

A molecular graph is a graph such that its vertices correspond to the atoms and edges to the bonds. The line graph $L(S(W_n))$ of a graph G is a graph where each vertex in $L(S(W_n))$ represents an edge in G. Two vertices in $L(S(W_n))$ are adjacent if and only if their corresponding edges in G are adjacent. The molecular graphs of subdivision graph and line graph of subdivision graph of wheel graph are represented in figure (1). Revan, reverse degree of end vertices of line graph of subdivision graph of wheel graph are obtained from degree of vertices. In a line graph of wheel graph, the 2-distance degree of an edge corresponds to how many edges are two steps away. The leap degree edge partition of line graph of wheel graph is given in table (3). Differential operators used in M/NM-polynomials computation are obtained from equations (13).

3. Results and Discussion

Reverse polynomials and indices of line graph of wheel graph

Theorem 1. First reverse Zagreb polynomial of $L(S(W_n))$ is $4nx^{2(n-4)} + nx^{n-5} + \frac{n(n-1)}{2}x^{-2}$.

Proof. This theorem is proved by using equations (1) and (14).

First reverse Zagreb polynomial of line graph of wheel graph

$CM_1(L(S(W_n,x))) = \sum_{\mathbf{uv\in E(G)}} x^{(c_u+c_v)}$
$= |E_{(n-4,n-4)}|x^{(n-4)+(n-4)} + |E_{(n-4,-1)}|x^{(n-4)+(-1)} + |E_{(-1,-1)}|x^{(-1)+(-1)}$
$= 4nx^{2(n-4)} + nx^{n-5} + \frac{n(n-1)}{2}x^{-2}$.

$CM_1(L(S(W_n))) = \frac{\partial CM_1(L(S(W_n,x)))}{\partial x}\Big|_{x=1} = \frac{\partial CM_1(4nx^{2(n-4)} + nx^{(n-5)} + \frac{n(n-1)}{2}x^{-2})}{\partial x}\Big|_{x=1}$
$= 4n(2n-9)$.

Theorem 2. Second reverse Zagreb polynomial of $L(S(W_n))$ is $4nx^{(n-4)^2} + nx^{(4-n)} + \frac{n(n-1)}{2}x$.

Proof. This theorem is proved by using equations (2) and (14).

Second reverse Zagreb polynomial of line graph of wheel graph

$CM_2(L(S(W_n,x))) = \sum_{\mathbf{uv\in E(G)}} x^{(c_u\times c_v)}$
$= |E_{(n-4,n-4)}|x^{(n-4)\times(n-4)} + |E_{(n-4,-1)}|x^{(n-4)\times(-1)} + |E_{(-1,-1)}|x^{(-1)\times(-1)}$
$= 4nx^{(n-4)^2} + nx^{(4-n)} + \frac{n(n-1)}{2}x$.

$CM_2(L(S(W_n))) = \frac{\partial CM_2(L(S(W_n,x)))}{\partial x}\Big|_{x=1} = \frac{\partial CM_2(4nx^{(n-4)^2} + nx^{(4-n)} + \frac{n(n-1)}{2}x)}{\partial x}\Big|_{x=1} = n(4n^2 - 33n + 68 + \frac{n-1}{2})$.

Theorem 3. First reverse hyper Zagreb polynomial of $L(S(W_n))$ is $4nx^{[2(n-4)]^2} + nx^{(n-5)^2} + \frac{n(n-1)}{2}x^4$.

Proof. First reverse hyper Zagreb polynomial of line graph of wheel graph

$CHM_1(L(S(W_n,x))) = \sum_{\mathbf{uv\in E(G)}} x^{(c_u+c_v)^2}$
$= |E_{(n-4,n-4)}|x^{[(n-4)+(n-4)]^2} + |E_{(n-4,-1)}|x^{[(n-4)+(-1)]^2} + |E_{(-1,-1)}|x^{[(-1)+(-1)]^2}$
$= 4nx^{[2(n-4)]^2} + nx^{(n-5)^2} + \frac{n(n-1)}{2}x^4$.

$CHM_1(L(S(W_n))) = \frac{\partial CHM_1(L(S(W_n,x)))}{\partial x}\Big|_{x=1} = \frac{\partial CHM_1\left(4nx^{[2(n-4)]^2} + nx^{(n-5)^2} + \frac{n(n-1)}{2}x^4\right)}{\partial x}\Big|_{x=1}$
$= n(17n^2 - 136n + 279)$.

Theorem 4. Second reverse hyper Zagreb polynomial of $L(S(W_n))$ is $4nx^{(n-4)^4}+nx^{(n-4)^2}+\frac{n(n-1)}{2}x$.

Proof. Second reverse hyper Zagreb polynomial of line graph of wheel graph

$CHM_2(L(S(W_n,x))) = \sum_{uv\in E(G)} x^{(c_u\times c_v)^2}$

$= |E_{(n-4,n-4)}|x^{[(n-4)\times(n-4)]^2} + |E_{(n-4,-1)}|x^{[(n-4)\times(-1)]^2} + |E_{(-1,-1)}|x^{[(-1)\times(-1)]^2}$

$= 4nx^{(n-4)^4}+nx^{(4-n)^2}+\frac{n(n-1)}{2}x$.

$CHM_2(L(S(W_n))) = \frac{\partial CHM_2(L(S(W_n,x)))}{\partial x}|_{x=1} = \frac{\partial CHM_2(4nx^{(n-4)^4}+nx^{(4-n)^2}+\frac{n(n-1)}{2}x)}{\partial x}|_{x=1}$

$= n[4(n-4)^4 + n^2 - 8n + \frac{n-1}{2}+16]$.

Revan polynomials and indices of line graph of wheel graph

Theorem 5. First Revan polynomial of $L(S(W_n))$ is $4nx^{2(n-2)} + nx^{n-1}+\frac{n(n-1)}{2}x^2$.

Proof. This theorem is proved by using equations (5) and (14).

First Revan polynomial of line graph of wheel graph is

$R_1(L(S(W_n,x))) = \sum_{uv\in E(G)} x^{(r_u+r_v)}$

$= |E_{(n-2,n-2)}|x^{(n-2)+(n-2)}+|E_{(n-2,1)}|x^{(n-2)+(1)} +|E_{(1,1)}|x^{(1)+(1)}$

$= 4nx^{2(n-2)} + nx^{n-1}+\frac{n(n-1)}{2}x^2$.

$R_1(L(S(W_n))) = \frac{\partial R_1(L(S(W_n,x)))}{\partial x}|_{x=1} = \frac{\partial R_1(4nx^{2(n-2)}+nx^{n-1}x+\frac{n(n-1)}{2}x^2)}{\partial x}|_{x=1} = 2n(5n-9)$.

Theorem 6. Second Revan polynomial of $L(S(W_n))$ is $4nx^{(n-2)^2}+nx^{n-1}+\frac{n(n-1)}{2}x$.

Proof. This theorem is proved by using equations (6) and (14).

$R_2(L(S(W_n,x))) = \sum_{uv\in E(G)} x^{(r_u\times r_v)}$

$= |E_{(n-2,n-2)}|x^{(n-2)\times(n-2)}+|E_{(n-2,1)}|x^{(n-1)\times(1)} +|E_{(1,1)}|x^{(1)\times(1)}$

$= 4nx^{(n-2)^2}+nx^{n-1}+\frac{n(n-1)}{2}x$.

$R_2(L(S(W_n))) = \frac{\partial R_2(L(S(W_n,x)))}{\partial x}|_{x=1} = \frac{\partial R_2(4nx^{(n-2)^2}+nx^{n-1}+\frac{n(n-1)}{2}x)}{\partial x}|_{x=1}$

$=4n^5 - 20n^4 + 32n^3 - \frac{31n^2+n}{2}$.

Theorem 7. First Revan hyper polynomial of $L(S(W_n))$ is $4nx^{[2(n-2)]^2} + nx^{(n-1)^2}+\frac{n(n-1)}{2}x^4$.

Proof. This theorem is proved by using equations (7) and (14).

$HR_1(L(S(W_n,x))) =\sum_{u\in E(G)} x^{(r_u+r_v)^2}$

$= |E_{(n-2,n-2)}|x^{[(n-2)+(n-2)]^2}+|E_{(n-2,1)}|x^{[(n-2)+1]^2} +|E_{(1,1)}|x^{(1+1)^2}$

$= 4nx^{[2(n-2)]^2} + nx^{(n-1)^2}+\frac{n(n-1)}{2}x^4$.

$HR_1(L(S(W_n))) = \frac{\partial HR_1(L(S(W_n,x)))}{\partial x}|_{x=1} = \frac{\partial HR_1(4nx^{[2(n-2)]^2}+nx^{(n-1)^2}+\frac{n(n-1)}{2}x^4)}{\partial x}|_{x=1}$

$= n(17x^2-64n+63)$.

Theorem 8. Second Revan hyper polynomial of $L(S(W_n))$ is $4nx^{(n-2)^4} + nx^{(n-2)^2}+\frac{n(n-1)}{2}x$.

Proof. This theorem is proved by using equations (8) and (14).

$HR_2(L(S(W_n,x))) =\sum_{u\in E(G)} x^{(r_u\times r_v)^2}$

$= |E_{(n-2,n-2)}|x^{[(n-2)\times(n-2)]^2}+|E_{(n-2,1)}|x^{[(n-2)\times1]^2}+|E_{(1,1)}|x^{(1\times1)^2}$

$= 4nx^{(n-2)^4} + nx^{(n-2)^2}+\frac{n(n-1)}{2}x$.

$HR_2(L(S(W_n))) = \frac{\partial HR_2(L(S(W_n,x)))}{\partial x}|_{x=1} = \frac{\partial HR_2(4nx^{(n-2)^4}+nx^{(n-2)^2}+\frac{n(n-1)}{2}x)}{\partial x}|_{x=1}$

$= n[4(n-2)^4+n^2-4n+4+\frac{n-1}{2}]$.

Leap polynomials and indices of line graph of wheel graph

Theorem 9. First leap Zagreb polynomial of $L(S(W_n))$ is $4nx^{8(n-1)} + nx^{7n-5}+\frac{n(n-1)}{2}x^{6n-2}$.

Proof. This theorem is proved by using equations (9) and (14).

$LM_1^*(L(S(W_n,x))) =\sum_{u\in E(G)} x^{[d_2(u)+d_2(v)]}$

$= |E_{(4(n-1),4(n-1))}|x^{[4(n-1)+4(n-1)]}+|E_{(4(n-1),(3n-1))}|x^{[4(n-1)+(3n-1)]} +|E_{((3n-1,3n-1))}|x^{[(3n-1)+(3n-1)]}$

$= 4nx^{8(n-1)} + nx^{7n-5}+\frac{n(n-1)}{2}x^{6n-2}$.

$LM_1^*(L(S(W_n))) = \frac{\partial LM_1^*(L(S(W_n,x)))}{\partial x}|_{x=1} = \frac{\partial LM_1^*(4nx^{8(n-1)}+nx^{7n-5}+\frac{n(n-1)}{2}x^{6n-2})}{\partial x}|_{x=1}$

$=n(35n-36+3n^2)$.

Theorem 10. Second leap Zagreb polynomial of $L(S(W_n))$ is $4nx^{[4(n-1)]^2} + nx^{[4(n-1)(3n-1)]}+\frac{n(n-1)}{2}x^{(3n-1)^2}$.

Proof. This theorem is proved by using equations (10) and (14).

$LM_2 (L(S(W_n,x))) =\sum_{u\in E(G)} x^{[d_2(u)\times d_2(v)]}$

$= |E_{(4(n-1),4(n-1))}|x^{[4(n-1)\times4(n-1)]} +|E_{(4(n-1),(3n-1))}|x^{[4(n-1)\times(3n-1)]} + |E_{((3n-1,3n-1))}|x^{[(3n-1)\times(3n-1)]}$

$= 4nx^{[4(n-1)]^2} + nx^{[4(n-1)(3n-1)]}+\frac{n(n-1)}{2}x^{(3n-1)^2}$.

$LM_2(L(S(W_n))) = \frac{\partial LM_2(L(S(W_n,x)))}{\partial x}|_{x=1} = \frac{\partial LM_2(4nx^{[4(n-1)]^2}+nx^{[4(n-1)(3n-1)]}+\frac{n(n-1)}{2}x^{(3n-1)^2})}{\partial x}|_{x=1}$

$=\frac{9n^4+137n^3-281n^2+135n}{2}$.

Theorem 11. First leap hyper Zagreb polynomial of $L(S(W_n))$ is $4nx^{[8(n-1)]^2} + nx^{(7n-5)^2}+\frac{n(n-1)}{2}x^{[2(3n-1)]^2}$.

Proof. This theorem is proved by using equations (11) and (14).

$LHM_1^*(L(S(W_n,x))) = \sum_{u\in E(G)} x^{[d_2(u)+d_2(v)]^2}$

$= |E_{(4(n-1),4(n-1))}| x^{[4(n-1)+4(n-1)]^2} + |E_{(4(n-1),(3n-1))}| x^{[4(n-1)+(3n-1)]^2} + |E_{(3n-1,3n-1)}| x^{[(3n-1)+(3n-1)]^2}$

$= 4nx^{[8(n-1)]^2} + nx^{(7n-5)^2} + \frac{n(n-1)}{2} x^{[2(3n-1)]^2}.$

$LHM_1^*(L(S(W_n))) = \frac{\partial LHM_1^*(L(S(W_n,x)))}{\partial x}|_{x=1} = \frac{\partial LHM_1^*(4nx^{[8(n-1)]^2}+nx^{(7n-5)^2}+\frac{n(n-1)}{2}x^{[2(3n-1)]^2})}{\partial x}|_{x=1}$

$= n(275n^2-568n+279+18n^3).$

Theorem 12. Second leap hyper Zagreb polynomial of $L(S(W_n))$ is $4nx^{[4(n-1)]^4} + nx^{[4(n-1)\times(3n-1)]^2} + \frac{n(n-1)}{2} x^{(3n-1)^4}$.

Proof. This theorem is proved by using equations (12) and (14).

$LHM_2(L(S(W_n,x))) = \sum_{u\in E(G)} x^{[d_2(u)\times d_2(v)]^2}$

$= |E_{(4(n-1),4(n-1))}| x^{[4(n-1)\times 4(n-1)]^2} + |E_{(4(n-1),(3n-1))}| x^{[4(n-1)\times(3n-1)]^2} + |E_{((3n-1,3n-1)}| x^{[(3n-1)\times(3n-1)]^2}$

$= 4nx^{[4(n-1)]^4} + nx^{[4(n-1)\times(3n-1)]^2} + \frac{n(n-1)}{2} x^{(3n-1)^4}.$

$LHM_2(L(S(W_n))) = \frac{\partial LHM_2(L(S(W_n,x)))}{\partial x}|_{x=1} = \frac{\partial LHM_2(4nx^{[4(n-1)]^4}+nx^{[4(n-1)\times(3n-1)]^2}+\frac{n(n-1)}{2}x^{(3n-1)^4})}{\partial x}|_{x=1}$

$= n(1024(n-1)^4 + 144n^4 + 352n^2 + 16 - 384n^3 - 128n + \frac{81n^5-108n^4+54n^3-12n^2+n-(3n-1)^4}{2}).$

M-polynomials and indices of line graph of wheel graph

Theorem 13. M_1-polynomial of $L(S(W_n))$ is $24nx^3y^3 + (3n+n^2)x^3y^n + n^2(n-1)x^ny^n$.

Proof. This theorem is proved by using equations (13).
M-polynomial of line graph of wheel graph

$M(G;x,y) = 4nx^3y^3 + nx^3y^n + \frac{n(n-1)}{2}x^ny^n.$

$D_xM(L(S(W_n;x,y)))$
$$= 12nx^3y^3 + 3nx^3y^n + \frac{n^2(n-1)}{2}x^ny^n.$$

$D_yM(L(S(W_n;x,y)))$
$$= 12nx^3y^3 + n^2x^3y^n + \frac{n^2(n-1)}{2}x^ny^n.$$

$(D_x + D_y)M(L(S(W_n;x,y))) = M_1(L(S(W_n;x,y)))$
$$= 24nx^3y^3 + (3n+n^2)x^3y^n + n^2(n-1)x^ny^n.$$

$M_1(L(S(W_n))) = M_1(L(S(W_n;x,y)))|_{x=y=1} = 27n + n^3.$

Theorem 14. M_2-polynomial of $L(S(W_n))$ is $36nx^3y^3 + 3n^2x^3y^n + \frac{n^3(n-1)}{2}x^ny^n$.

Proof. This theorem is proved by using equations (13).
M-polynomial of line graph of wheel graph $M(G;x,y) = 4nx^3y^3 + nx^3y^n + \frac{n(n-1)}{2}x^ny^n$.

$D_xM(L(S(W_n;x,y)))$
$$= 12nx^3y^3 + 3nx^3y^n + \frac{n^2(n-1)}{2}x^ny^n.$$

$D_yM(L(S(W_n;x,y)))$
$$= 12nx^3y^3 + n^2x^3y^n + \frac{n^2(n-1)}{2}x^ny^n.$$

$(D_x \times D_y)M(L(S(W_n;x,y))) = M_2(L(S(W_n;x,y)))$
$$= 36nx^3y^3 + 3n^2x^3y^n + \frac{n^3(n-1)}{2}x^ny^n.$$

$M_2(L(S(W_n))) = M_2(L(S(W_n;x,y)))|_{x=y=1} = n(36 + 3n + \frac{n^3-n^2}{2}).$

Theorem 15. HM_1-polynomial of $L(S(W_n))$ is $144nx^3y^3 + (n^3+6n^2+9n)x^3y^n + [2n^3(n-1) + \frac{n^3(n-1)}{2}]x^ny^n$.

Proof. This theorem is proved by using equations (13).

$M(L(S(W_n;x,y))) = 4nx^3y^3 + nx^3y^n + \frac{n(n-1)}{2}x^ny^n.$

$D_xM(L(S(W_n;x,y)))$
$$= 12nx^3y^3 + 3nx^3y^n + \frac{n^2(n-1)}{2}x^ny^n.$$

$D_yM(L(S(W_n;x,y)))$
$$= 12nx^3y^3 + n^2x^3y^n + \frac{n^2(n-1)}{2}x^ny^n.$$

$(D_x \times D_y)M(L(S(W_n;x,y)))$
$$= 36nx^3y^3 + 3n^2x^3y^n + \frac{n^3(n-1)}{2}x^ny^n.$$

$D_x^2M(L(S(W_n;x,y)))$
$$= 36nx^3y^3 + 9nx^3y^n + \frac{n^3(n-1)}{2}x^ny^n.$$

$D_y^2M(L(S(W_n;x,y)))$
$$= 36nx^3y^3 + n^3x^3y^n + \frac{n^3(n-1)}{2}x^ny^n.$$

$(D_x + D_y)^2HM(L(S(W_n;x,y))) = HM_1(L(S(W_n;x,y))) = 144nx^3y^3 + (n^3+6n^2+9n)x^3y^n + [2n^3(n-1) + \frac{n^3(n-1)}{2}]x^ny^n.$

$HM_1(L(S(W_n))) = HM_1(L(S(W_n;x,y)))|_{x=y=1}$

$= n\left(153 - n^2 + 6n + 2n^3 + \frac{n^3-n^2}{2}\right).$

Theorem 16. HM_2-polynomial of $L(S(W_n))$ is $324nx^3y^3 + 3n^3x^3y^n + \frac{n^5(n-1)}{2}x^ny^n$.

Proof. This theorem is proved by using equations (13).

$M(L(S(W_n;x,y))) = 4nx^3y^3 + nx^3y^n + \frac{n(n-1)}{2}x^ny^n.$

$D_xM(L(S(W_n;x,y)))$
$$= 12nx^3y^3 + 3nx^3y^n + \frac{n^2(n-1)}{2}x^ny^n.$$

$D_yM(L(S(W_n;x,y)))$
$$= 12nx^3y^3 + n^2x^3y^n + \frac{n^2(n-1)}{2}x^ny^n.$$

$D_x^2M(L(S(W_n;x,y)))$
$$= 36nx^3y^3 + 3nx^3y^n + \frac{n^3(n-1)}{2}x^ny^n.$$

$(D_x \times D_y)^2M(L(S(W_n;x,y))) = HM_2(L(S(W_n;x,y)))$
$$= 324nx^3y^3 + 3n^3x^3y^n + \frac{n^5(n-1)}{2}x^ny^n.$$

$HM_2(L(S(W_n))) = HM_2(L(S(W_n;x,y)))|_{x=y=1} = n(324n + 3n^2 + \frac{n^5-n^4}{2}).$

NM-polynomials and indices of line graph of wheel graph

Theorem 17. NM_1-polynomial of $L(S(W_n))$ is

$$2(n-1)(2n+8)x^{(2n+8)}y^{(2n+8)}+2(n-1)[(2n+8)+n(n-2)+8]x^{(2n+8)}y^{n(n-2)+8}+(n-1)(n-2)[n(n-2)+8]x^{n(n-2)+8}\,y^{n(n-2)+8}.$$

Proof. This theorem is proved by using equations (13).

$$NM(L(S(W_n;x,y)))=(n-1)x^{(2n+8)}y^{(2n+8)}+2(n-1)x^{(2n+8)}y^{n(n-2)+8}+\frac{(n-1)(n-2)}{2}x^{n(n-2)+8}\,y^{n(n-2)+8}.$$

$$D_xNM(L(S(W_n;x,y)))=(n-1)(2n+8)x^{(2n+8)}y^{(2n+8)}+2(n-1)(2n+8)x^{(2n+8)}y^{n(n-2)+8}+\frac{(n-1)(n-2)}{2}2[n(n-2)+8]x^{n(n-2)+8}y^{n(n-2)+8}.$$

$$D_yNM(L(S(W_n;x,y)))=(n-1)(2n+8)x^{(2n+8)}y^{(2n+8)}+2(n-1)(n(n-2)+8)x^{(2n+8)}y^{n(n-2)+8}+\frac{(n-1)(n-2)}{2}2[n(n-2)+8]x^{n(n-2)+8}\,y^{n(n-2)+8}$$

$$(D_x+D_y)NM(L(S(W_n;x,y)))=NM_1(L(S(W_n;x,y)))=2(n-1)(2n+8)x^{(2n+8)}y^{(2n+8)}+2(n-1)[(2n+8)+n(n-2)+8]x^{(2n+8)}y^{n(n-2)+8}+(n-1)(n-2)2[n(n-2)+8]x^{n(n-2)+8}\,y^{n(n-2)+8}.$$

$NM_1(L(S(W_n)))=NM_1(L(S(W_n;x,y)))|_{x=y=1}=(n-1)(8+14n+n^3-3n^2)$.

Theorem 18. NM_2-polynomial of $L(S(W_n))$ is

$$(n-1)(2n+8)^2x^{(2n+8)}y^{(2n+8)}+2(n-1)(2n+8)^2(n(n-2)+8)^2x^{(2n+8)}y^{n(n-2)+8}+\frac{(n-1)(n-2)}{2}(n(n-2)+8)^2x^{n(n-2)+8}\,y^{n(n-2)+8}.$$

Proof. This theorem is proved by using equations (13).

$$NM(L(S(W_n;x,y)))=(n-1)x^{(2n+8)}y^{(2n+8)}+2(n-1)x^{(2n+8)}y^{n(n-2)+8}+\frac{(n-1)(n-2)}{2}x^{n(n-2)+8}\,y^{n(n-2)+8}.$$

$$D_xNM(L(S(W_n;x,y)))=(n-1)(2n+8)x^{(2n+8)}y^{(2n+8)}+2(n-1)(2n+8)x^{(2n+8)}y^{n(n-2)+8}+\frac{(n-1)(n-2)}{2}2[n(n-2)+8]x^{n(n-2)+8}y^{n(n-2)+8}.$$

$$D_yNM(L(S(W_n;x,y)))=(n-1)(2n+8)x^{(2n+8)}y^{(2n+8)}+2(n-1)(n(n-2)+8)x^{(2n+8)}y^{n(n-2)+8}+\frac{(n-1)(n-2)}{2}2[n(n-2)+8]x^{n(n-2)+8}\,y^{n(n-2)+8}$$

$$(D_x\times D_y)NM(L(S(W_n;x,y)))=NM_2(L(S(W_n;x,y)))=(n-1)(2n+8)^2x^{(2n+8)}y^{(2n+8)}+2(n-1)(2n+8)^2(n(n-2)+8)^2x^{(2n+8)}y^{n(n-2)+8}+\frac{(n-1)(n-2)}{2}(n(n-2)+8)^2x^{n(n-2)+8}\,y^{n(n-2)+8}.$$

$NM_2(L(S(W_n)))=NM_2(L(S(W_n;x,y)))|_{x=y=1}=\frac{n^6-8n^5+48n^4-112n^3+312n^2-64n-128}{2}$.

Theorem 19. NHM_1-polynomial of $L(S(W_n))$ is

$$(n-1)[2(2n+8)]^2x^{(2n+8)}y^{(2n+8)}+2(n-1)[(2n+8)+n(n-2)+8]^2x^{(2n+8)}y^{n(n-2)+8}+(n-1)(n-2)[n(n-2)+8]^2x^{n(n-2)+8}\,y^{n(n-2)+8}.$$

Proof. This theorem is proved by using equations (13).

$$NM(L(S(W_n;x,y)))=(n-1)x^{(2n+8)}y^{(2n+8)}+2(n-1)x^{(2n+8)}y^{n(n-2)+8}+\frac{(n-1)(n-2)}{2}x^{n(n-2)+8}\,y^{n(n-2)+8}.$$

$$D_xNM(L(S(W_n;x,y)))=(n-1)(2n+8)x^{(2n+8)}y^{(2n+8)}+2(n-1)(2n+8)x^{(2n+8)}y^{n(n-2)+8}+\frac{(n-1)(n-2)}{2}(n(n-2)+8)x^{n(n-2)+8}\,y^{n(n-2)+8}.$$

$$D_yNM(L(S(W_n;x,y)))=(n-1)(2n+8)x^{(2n+8)}y^{(2n+8)}+2(n-1)[n(n-2)+8]x^{(2n+8)}y^{n(n-2)+8}+\frac{(n-1)(n-2)}{2}(n(n-2)+8)x^{n(n-2)+8}\,y^{n(n-2)+8}.$$

$$(D_x+D_y)^2NM(L(S(W_n;x,y)))=NHM_1(L(S(W_n;x,y)))=4(n-1)(2n+8)(5+n)x^{(2n+8)}y^{(2n+8)}+2(n-1)(160+26x^{(2n+8)}y^{n(n-2)+8}+(n-1)(n-2)[n(n-2)+8]^2x^{n(n-2)+8}\,y^{n(n-2)+8}.$$

$NHM_1(L(S(W_n)))=NHM_1(L(S(W_n;x,y)))|_{x=y=1}$
$=(n-1)(8n^2+256n+640+n^5+28n^3-4n^4)$.

Theorem 20. NHM_2-polynomial of $L(S(W_n))$ is

$$(n-1)(2n+8)^4x^{(2n+8)}y^{(2n+8)}+2(n-1)[(2n+8)(n(n-2))+8]^2x^{(2n+8)}y^{n(n-2)+8}+\frac{(n-1)(n-2)}{2}[n(n-2)+8]^44x^{n(n-2)+8}\,y^{n(n-2)+8}.$$

Proof. This theorem is proved by using equations (13).

$$NM(L(S(W_n;x,y)))=(n-1)x^{(2n+8)}y^{(2n+8)}+2(n-1)x^{(2n+8)}y^{n(n-2)+8}+\frac{(n-1)(n-2)}{2}x^{n(n-2)+8}\,y^{n(n-2)+8}.$$

$$D_xNM(L(S(W_n;x,y)))=(n-1)(2n+8)x^{(2n+8)}y^{(2n+8)}+2(n-1)(2n+8)x^{(2n+8)}y^{n(n-2)+8}+\frac{(n-1)(n-2)}{2}[n(n-2)+8]x^{n(n-2)+8}\,y^{n(n-2)+8}.$$

$$D_yNM(L(S(W_n;x,y))) = (n-1)(2n+8)x^{(2n+8)}y^{(2n+8)} + 2(n-1)[n(n-2)+8]x^{(2n+8)}y^{n(n-2)+8} + \frac{(n-1)(n-2)}{2}[n(n-2)+8]x^{n(n-2)+8}y^{n(n-2)+8}.$$

$$(D_x \times D_y)^2NM(L(S(W_n;x,y))) = NHM_2(L(S(W_n;x,y))) = (n-1)(2n+8)^4x^{(2n+8)}y^{(2n+8)} + 2(n-1)[(2n+8)(n(n-2))+8]^2x^{(2n+8)}y^{n(n-2)+8} + \frac{(n-1)(n-2)}{2}[n(n-2)+8]^4 4x^{n(n-2)+8}y^{n(n-2)+8}.$$

$$NHM_2(L(S(W_n))) = NHM_2(L(S(W_n;x,y)))|_{x=y=1}$$
$$=(n-1)(2n+8)^4 + 2(n-1)((2n+8)n(n-2)+8)^2 + \frac{(n-1)(n-2)}{2}(n(n-2)+8)^4.$$

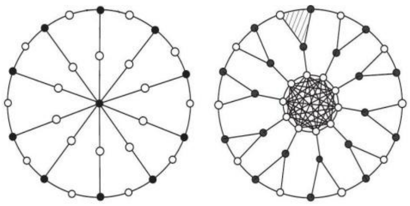

Figure 1: The subdivision graph of the wheel W_n and the line graph of subdivision graph of the wheel $L(S(W_n))$.

Table 1: The reverse degree edge partition of $L(S(W_n))$.

Reverse degree	(n-4, n-4)	(n-4, -1)	(-1, -1)
Number of edges	4n	n	$\frac{n(n-1)}{2}$

Table 2: The Revan degree edge partition of $L(S(W_n))$.

Revan degree	(n-2, n-2)	(n-2,1)	(1,1)
Number of edges	4n	n	$\frac{n(n-1)}{2}$

Table 3: The leap degree edge partition of $L(S(W_n))$

Degree	$(d_2(3), d_2(3))$	$(d_2(3), d_2(n))$	$(d_2(n), d_2(n))$
Leap degree	(4(n-1), 4(n-1))	(4(n-1), 3n-1)	(3n-1, 3n-1)
Number of edges	4n	n	$\frac{n(n-1)}{2}$

4. Conclusion

Reverse, Revan, leap polynomials, hyper-polynomials, M-polynomials, NM-polynomials and their indices are studied for line graph of subdivision graph of wheel graph.

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