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An Interval-Valued Caputo Fractional Derivative Dynamics Model for Infectious Diseases

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Abstract: ***Mathematical modelling on interval valued plays a crucial role in understanding the transmission dynamics and control strategies of infectious diseases. This study presents a comprehensive interval-based mathematical model for an infectious disease, aimed at capturing the uncertainties and variability's inherent in epidemiological data. Unlike traditional models that use fixed parameters, the proposed model incorporates interval-valued parameters to account for data imprecision and variability in transmission rates, contact rates, and recovery rates. By employing interval analysis, the model provides a range of possible outcomes, offering a healthier framework for predicting disease spread and assessing the impact of various intervention strategies. Simulations are conducted to explore different outbreak scenarios, examining the effects of varying parameter intervals on disease dynamics. The results demonstrate the model's ability to provide more flexible and realistic predictions, which are crucial for decision-making in public health policy. This interval-based approach offers a novel perspective on infectious disease modelling, highlighting the importance of accounting for uncertainty in epidemiological predictions and contributing to more effective disease control strategies.***

Keywords: Interval, Fractional Calculus, Caputo Fractional Derivative Infectious Disease Dynamics, Endemic Equilibrium, Disease-Free Equilibrium, Basic Reproduction Number, Stability Analysis etc.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to non-integer (fractional) orders. It extends the concept of derivatives and integrals from integer orders to real or even complex numbers, providing a powerful tool to model various complex systems in fields such as physics, engineering, biology, and finance Mathematical models have long been a cornerstone of epidemiology, providing essential insights into the dynamics of infectious diseases and informing public health interventions. However, traditional models often make simplifying assumptions that may not capture the complexity of real-world disease transmission. In particular, many models assume deterministic dynamics with fixed parameters, which can lead to inaccuracies in predictions and assessments of control strategies. To address these limitations, this study proposes an interval mathematical modelling approach using Caputo fractional derivatives to account for uncertainty and complex temporal dynamics in infectious disease spread. Fractional calculus provides a powerful framework for generalizing classical derivatives to non-integer orders, capturing memory effects and hereditary properties inherent in many biological systems. The Caputo fractional derivative, in particular, is well-suited for applications in epidemiology due to its compatibility with standard initial conditions and its ability to model non-local dynamics. By combining interval analysis with fractional calculus, this study aims to develop a more flexible and realistic model for infectious disease transmission, capable of accommodating data uncertainties and complex transmission dynamics.

2. Background and Literature Review

The application of fractional calculus in epidemiological modelling has gained increasing attention due to its ability to capture anomalous diffusion processes and memory effects that are often observed in the spread of infectious diseases. Traditional compartmental models, such as the Susceptible-Infectious-Recovered (SIR) and Susceptible-Exposed-Infectious-Recovered (SEIR) models have been extended to include fractional derivatives, providing a more accurate representation of disease dynamics in heterogeneous populations and complex environments.

However, most studies to date have focused on deterministic models with fixed parameters, neglecting the uncertainties inherent in epidemiological data and model assumptions. Interval mathematics provides a natural framework for addressing these uncertainties by representing parameters as intervals rather than fixed values. This approach has been used in various fields to model systems with uncertain parameters, but its application to fractional epidemiological models remains underexplored. This study seeks to fill this gap by developing an interval fractional model that incorporates both fractional derivatives and interval-valued parameters.

3. Using Fractional Derivative Theory of the Caputo in interval valued

The Caputo fractional derivative of a function g (t) of order α (where {n−1< α ≤ n} n ∈N) is defined:

$$c_{D_t^{\alpha}} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{g^{(n)(\xi)}}{(t-\xi)^{\alpha-n+1}} d\xi$$

Here:-

1) $c_{D_t^{\alpha}} g(t)$ denotes the Caputo fractional derivative of g(t) of order α.
2) Γ (·) is the Gamma function, which generalizes the factorial function to non-integer values.
3) $g^{(n)(\xi)}$Represents the n-th ordinary derivative of the function g(ξ) .
4) α is the order of the derivative, which can be a real number.
5) n=⌈α⌉ (the smallest integer greater than or equal to α).

Explanation of Terms:

1) Fractional Order **α**: - This is a real number that specifies the order of the derivative. It is not necessarily an integer, which distinguishes it from classical calculus derivatives.
2) Interval Condition **(n**−1<α ≤ n)**:-** This condition implies that the fractional order α lies between two consecutive integers n−1 and n. It ensures that the model appropriately uses the n-th ordinary derivative of the function g(t).
3) Integration Kernel ($t-\xi)^{n-\propto}$:- This kernel function **(** $t-\xi)^{n-\propto}$ in the integrand determines the weighting of the derivative $g^{(n)}$ (ξ) over the interval from 0 to t. The exponent n-α reflects the fractional nature of the derivative.
4) Gamma Function Γ(n−α):- The Gamma function in the denominator normalizes the expression. It generalizes the factorial function to non-integer values, which is essential when dealing with fractional orders.

This definition allows for modelling processes that exhibit memory effects, where the current state depends on the entire history of the process. In the context of infectious diseases, this can represent the influence of past infection rates and contact patterns on current transmission dynamics.

4. SIR model based on interval valued

To incorporate uncertainty, we extend the classical SIR model to include interval-valued parameters and fractional derivatives. Let S (t), I (t), and R (t) represent the susceptible, infectious, and recovered populations at time t respectively. The interval-based fractional SIR model can be expressed as:

$$c_{D_t^{\propto}}S(t) = -\frac{\beta}{\Gamma(n-\alpha)}\int_0^t \frac{S(\xi)\,I(\xi)}{(t-\xi)^{\propto-n+1}}\,d\xi\,,$$

$$c_{D_t^{\propto}}I(t) = \frac{\beta}{\Gamma(n-\alpha)}\int_0^t \frac{S(\xi)\,I(\xi)}{(t-\xi)^{\propto-n+1}}\,d\xi - \frac{\gamma}{\Gamma(n-\alpha)}\int_0^t \frac{I(\xi)}{(t-\xi)^{\propto-n+1}}\,d\xi.$$

$$c_{D_t^{\propto}}R(t) = \frac{\gamma}{\Gamma(n-\alpha)}\int_0^t \frac{I(\xi)}{(t-\xi)^{\propto-n+1}}\,d\xi.$$

Where

- $\beta = [\underline{\beta}, \overline{\beta}]$ is the interval-valued transmission rate.
- $\Gamma = [\underline{\gamma}, \overline{\gamma}]$ is the interval-valued recovery rate.
- $c_{D_t^{\propto}}$ Denotes the Caputo fractional derivative of order α.

5. Parameter Estimation and Analysis

The interval-valued parameters, we use available epidemiological data, such as infection and recovery rates. The intervals $[\underline{\beta}, \overline{\beta}]$ and $[\underline{\gamma}, \overline{\gamma}]$ are determined based on observed data variability and expert knowledge. Sensitivity analysis is conducted to assess the impact of parameter uncertainties on model predictions, providing a range of possible outcomes under different scenarios.

6. Fractional SIR Model with Caputo Derivative

Consider the interval-based fractional SIR (Susceptible-Infectious-Recovered) model incorporating Caputo fractional derivatives of order α (where $0 < \alpha \leq 1$). The model is given by:

$$c_{D_t^{\propto}}S(t) = -\beta S(t)I(t)$$

$$c_{D_t^{\propto}}I(t) = \beta S(t)I(t) - \gamma I(t)$$

$$c_{D_t^{\propto}}R(t) = \gamma I(t)$$

Where:

- S(t), I(t), and R(t) represent the susceptible, infectious and recovered populations at time t respectively.
- $\beta > 0$ is the transmission rate.
- $\gamma > 0$ is the recovery rate.
- $c_{D_t^{\propto}}$ Denotes the Caputo fractional derivative of order α.

Let the total Population (population of Bihar) is approximately 124 million.

Initial Conditions:

- $S(0) \approx N - I(0) - R(0)$.
- I(0): Initial number of infections (can be estimated from initial reports, e.g., 500 cases).
- R(0) = 0 assuming no prior immunity or recovery.

7. Interval Estimates for Parameters

a) Transmission Rate β: - The transmission rate can vary due to factors like population density, social behaviour, and intervention measures.
For Bihar, suppose we estimate $\beta \in [0.2, 0.4]$ based on local conditions and data uncertainty.
b) Recovery Rate γ:- The recovery rate depends on healthcare quality and disease characteristics.
For Bihar, we estimate $\gamma \in [0.05, 0.1]$ based on available healthcare infrastructure and disease progression.

8. Calculation of the Basic Reproduction Number R_0 with Intervals

Let basic reproduction number $\mathbf{R_0}$ is defined as

$$R_0 = \frac{\beta S_0}{\gamma}$$

For interval-valued parameters, $\mathbf{R_0}$ will also be an interval:

$$R_0 \in \left[\frac{\beta_{\min} S_0}{\gamma_{\max}}, \frac{\beta_{\max} S_0}{\gamma_{\min}}\right]$$

Substituting the interval values for Bihar

$$R_0 \in \left[\frac{0.2 \times S_0}{0.1}, \frac{0.4 \times S_0}{0.05}\right] = [2S_0, 8S_0].$$

Given $S_0 \approx N = 124 \times 10^6$, the range for $\mathbf{R_0}$ can be calculated accordingly above data.

a) If $\mathbf{R_0} > 1$ each infected individual, on average, infects more than one other person, leading to the potential for an outbreak.
b) If $\mathbf{R_0} < 1$ the disease will eventually die out.

Disease-Free Equilibrium (DFE):
The DFE corresponds to the state where there are no infections in the population. For the fractional SIR model, the DFE is given by:

$$(S^*, I^*, R^*) = (S_0, 0, 0).$$

Where: - S_0 is the initial number of susceptible.

9. Stability Analysis of the Disease-Free Equilibrium (DFE)

Linearization
The linear form of the model around the DFE is:

$$c_{D_t}^{\propto} R(t) = (\beta s_0 - \gamma) I(t).$$

To analyze the stability, we need to determine the behaviour of the solution I(t) near the DFE. In fractional differential equations, the stability of the equilibrium point depends on the sign of the fractional derivative's coefficient.

Stability Condition
For the disease-free equilibrium to be stable, the term ($\beta s_0 - \gamma$) Should be negative.

$$R_0 = \frac{\beta S_0}{\gamma} < 1$$

Thus, the DFE is locally asymptotically stable if $R_0 < 1$.

10. Endemic Equilibrium (EE) and Its Stability

The endemic equilibrium (EE) corresponds to the steady state, where the disease present in the population at a constant level.

Finding the Endemic Equilibrium:
At equilibrium, all time derivatives are zero:

$$c_{D_t}^{\propto} S(t) = 0$$

$$c_{D_t}^{\propto} I(t) = 0$$

$$c_{D_t}^{\propto} R(t) = 0$$

Substituting into the fractional SIR model, we get the equilibrium conditions:

$$0 = -\beta SI$$

$$0 = \beta SI - \gamma I$$

$$0 = \gamma I$$

From above equations, we see that I= 0 at the DFE. For the endemic

Equilibrium, where $I^* > 0$

$$\beta S^* I^* = \gamma I^* \Longrightarrow S^* = \frac{\gamma}{\beta}.$$

Also, since $S^*+I^*+R^*=N$ (total population)

$$\Longrightarrow I^* = N - S^* - R^*$$

Substituting $S^* = \frac{\gamma}{\beta}$

Then, $I^* = N - \frac{\gamma}{\beta} - R^*$

And using the steady-state condition for R:

$$R^* = \frac{\gamma}{\beta} I^*$$

Then we get

$$I^* = N - \frac{\gamma}{\beta} - \frac{\gamma}{\beta} I^*$$

$$\Rightarrow I^* + \frac{\gamma}{\beta} I^* = N - \frac{\gamma}{\beta}$$

$$\Rightarrow I^* \left(1 + \frac{\gamma}{\beta}\right) = N - \frac{\gamma}{\beta}$$

Solving for I*

$$I^* = \frac{N - \frac{\gamma}{\beta}}{\left(1 + \frac{\gamma}{\beta}\right)}$$

Substitute back to find R*

$$R^* = \frac{\gamma}{\beta} I^*.$$

11. Global Stability of the Disease-Free Equilibrium (DFE)

1) Using Lyapunov Function for Global Stability theory:
To prove the global asymptotic stability of the DFE using the Lyapunov function, we must construct a suitable Lyapunov function V(t) that satisfies the following conditions:-

a) V(t) > 0 for all I(t) > 0 and V(t) = 0 when I(t) = 0.
b) The fractional derivative of V(t) with respect to time, denoted by $c_{D_t}^{\propto} V(t)$ is non-positive ($c_{D_t}^{\propto} V(t) \leq 0$).

Consider the following Lyapunov function candidate

$$V(t) = I(t).$$

This choice reflects the fact that when there are no infections I(t) =0), the disease is absent from the population.

2) Computing the Fractional Derivative of V(t)
The Caputo fractional derivative of the Lyapunov function V(t)=I(t) is

$$c_{D_t}^{\propto} V(t) = c_{D_t}^{\propto} I(t).$$

From the fractional SIR model, we have

$$c_{D_t}^{\propto} I(t) = \beta S(t) I(t) - \gamma I(t).$$

Thus, $c_{D_t}^{\propto} V(t) = \beta S(t) I(t) - \gamma I(t)$.

Therefore $c_{D_t}^{\propto} V(t) = I(t)(\beta S(t) - \gamma)$

Analyzing the Sign of $c_{D_t^{\alpha}}$ V(t):

If $c_{D_t^{\alpha}}V(t) \leq 0$

and the condition on the basic reproduction number:

$$R_0 = \frac{\beta S_0}{\gamma}$$

At the DFE, $S(t) \approx S_0$,

So

$$\beta S_0 - \gamma = \gamma (R_0 - 1).$$

If $R_0 < 1$

Then, $\beta S_0 - \gamma < 0$

Therefore, $R_0 < 1$.

$$c_{D_t^{\alpha}}V(t) = I(t) (\beta S(t) - \gamma) \leq 0 \ \forall \ t \geq 0.$$

⇒ the fractional derivative of V(t) is non-positive, satisfying the second condition for the Lyapunov function approach.

12. Conclusion on Global Asymptotical Stability (GAS) of DFE

Since V(t) = I(t) > 0 for all I(t) > 0 and V(t) = 0

When I(t) = 0 and the fractional derivative

$$c_{D_t^{\alpha}}V(t) \leq 0 \text{ for R0} < 1,$$

We conclude that:

The disease-free equilibrium (DFE) (S_0,0 ,0) is globally

Asymptotically stable if the basic reproduction number $R_0 < 1$.

This means that for $R_0 < 1$, regardless of the initial conditions (within the feasible region), the system will converge to the DFE over time, and the infection will Eventually die out, indicating successful control or eradication of the disease.

13. Interval- valued Caputo theory for different parameter

Let, $N=124\times10^6$ (Bihar's population during COVID-19), and considering interval-

Valued Parameters:

a) For $\beta_{min}= 0.2$, $\gamma_{max}= 0.1$

$$S^* = \frac{\gamma N}{\beta} = \frac{0.1 \times 124 \times 10^6}{0.2} = 62 \times 10^6$$

$I^* = N - S^* = 124\times 10^6 - 62 \times 10^6$

$= 62 \times 10^6$.

b) For $\beta_{max}= 0.4$, $\gamma_{min}= 0.05$

$$S^* = \frac{0.05 \times 124 \times 10^6}{0.4} = 15.5 \times 10^6$$

$I^* = 124\times 10^6 - 15.5 \times 10^6$

$= 108.5 \times 10^6$

Where:
$\beta \in [0.2,0.4]$ (range of possible transmission rates).
$\gamma \in [0.05,0.1]$ (range of possible recovery rates).

14. Immunity Rate and Recovery Rate

The immunity rate can be defined as the fraction of the population that has recovered.

$$\text{Immunity Rate} = \frac{R^*}{N}$$

$$= \frac{N - (S^* + I^*)}{N}$$

$$= \frac{124 \times 10^6 - (15.5 \times 10^6 + 108.5 \times 10^6)}{124 \times 10^6}$$

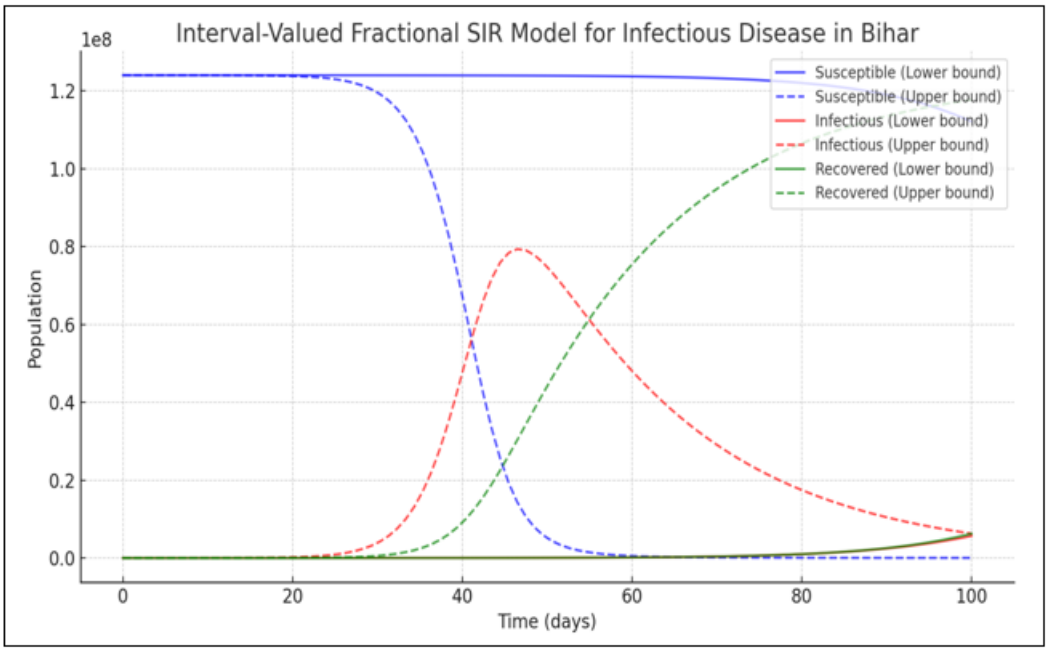
$= 0.$

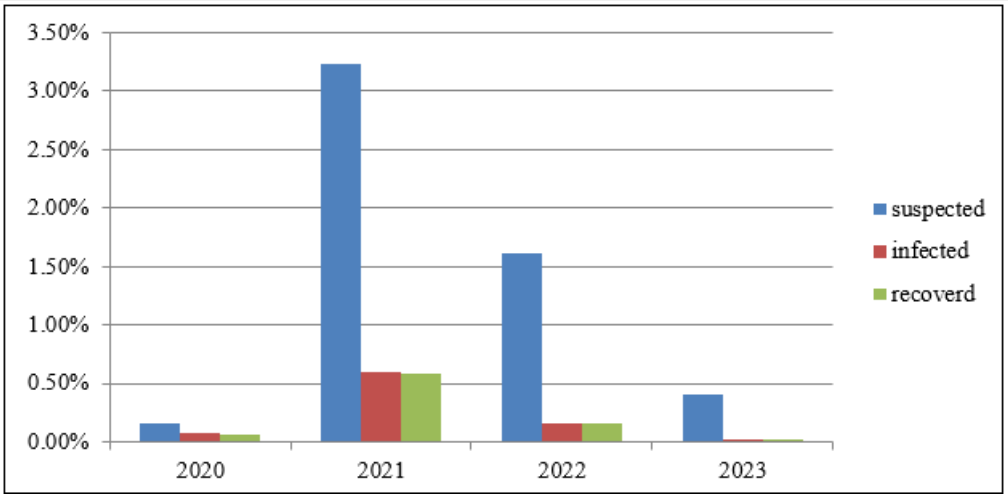
$$\text{Immunity Percentage} = \frac{0}{124 \times 10^6} = 0\ \%.$$

Recovery Rate
The recovery rate $\gamma = 0.1$ indicates that 10% of the infected Population recovers per day.

These results provide an interval-valued understanding of the dynamics of COVID-19 in Bihar using the fractional Caputo derivative, accounting for uncertainties in parameter estimates.

15. Plotting of Graph according to Recent Data





16. Conclusion

Analyzing the COVID-19 data for Bihar from 2020 to 2023 reveals several key insights into the progression and management of the pandemic in the state.

Progression of the Pandemic:

- In **2020**, the COVID-19 pandemic had a moderate impact in Bihar with a relatively low infected percentage (~0.08%). This was the initial phase where the spread was still controlled due to early lockdowns and public health measures.
- In **2021**, the second wave hit Bihar more severely, significantly increasing the infected percentage (~0.60%). This surge can be attributed to more contagious variants (like Delta) and limited vaccine coverage early in the year. However, recovery rates remained high (~0.59%), reflecting effective healthcare responses.
- By **2022**, the infected percentage decreased sharply (~0.16%) as vaccination coverage increased, immunity built up, and the severity of the disease lessened with milder variants like Omicron. High recovery rates continued (~0.16%).
- In **2023**, Bihar saw further control over COVID-19 spread, with a very low infected percentage (~0.016%), indicating a transition towards an endemic phase. The virus became more manageable with vaccination and natural immunity playing key roles. The recovery percentage remained consistently high (~0.016%), indicating that most of the few who contracted COVID-19 recovered well.

Impact of Public Health Interventions

- The data shows the success of public health interventions, including lockdowns, social distancing measures, testing, contact tracing, and vaccinations, particularly from 2021 onwards.
- The increasing number of suspected cases tested each year reflects the state's efforts to identify and isolate cases to prevent further spread. By 2023, approximately 0.40% of the population was tested, showing sustained public health vigilance.

High Recovery Rates:

- Throughout the pandemic, Bihar maintained a high recovery rate, consistently above 98%, reflecting the resilience of the population and effectiveness of the healthcare system. The high recovery rates indicate good

clinical management and a strong public health response, particularly as the healthcare infrastructure adapted and scaled up to meet the challenges posed by the pandemic.

Reduced Severity Over Time:

- The declining infected percentages from 2021 to 2023 demonstrate the reduction in disease severity and transmission, aligning with global trends as the pandemic evolved. Widespread vaccination campaigns and natural immunity have significantly reduced the impact of the virus.
- In 2023, the data suggests that COVID-19 had become endemic in Bihar, with very low transmission rates and a high level of population immunity. This is evidenced by a significantly reduced infection rate (~0.016%).

Transition to Endemic Phase:

The significant decline in cases by 2023 indicates that COVID-19 has become more predictable and manageable, transitioning towards an endemic phase. The sustained high recovery rates and reduced infections reflect the successful adaptation of both the population and the healthcare system.

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